The Circle

Action on THH

Recall: $B : \mathrm{Alg}^{gp}_{\mathbb{E}_1}(\mathcal{S}) \underset{\Omega}{\overset{\sim}{\rightleftarrows}} \mathcal{S}^{\geq 1}_*$

$\mathbb{E}_1$-group

connected spaces

(May's recognition principle)

coincide w/ $NB : \mathrm{Group} \to \mathcal{S}_*$

<u>Defn</u> $G \in \mathrm{Alg}^{gp}_{\mathbb{E}_1}(\mathcal{S})$

$\mathcal{C} \in \mathrm{Cat}_\infty$.

Be careful! G is no longer discrete!

define the cat of objects in $\mathcal{C}$ w/ G-action as

$$\mathrm{Fun}(BG, \mathcal{C})$$

as $BG \in \mathcal{S}_*$, one can assoc a $* \to BG \to \mathcal{C}$, i.e. a "canonical" object in $\mathcal{C}$.

<u>Defn</u> homotopy orbit / fixed pt

<u>special case</u> in $\mathcal{S}$, if X has the trivial action,

$$X_{hG} \simeq X \times BG$$

$$X^{hG} \simeq \mathrm{Map}(BG, X)$$

$$\begin{array}{c} X \times BG \\ \downarrow \\ BG \longrightarrow \mathcal{S} \end{array}$$

$X \times BG /$ cocartesian edges $= X \times BG$

cocartesian section $BG \to X \times BG$

$= \mathrm{Map}(BG, X)$.

Prop $\mathrm{Fun}(BS^1, \mathcal{D}(\mathbb{Z}))$

$\simeq \mathrm{Mod}_A(\mathcal{D}(\mathbb{Z}))$ not a monoidal equiv.

$$A = \Lambda_{\mathbb{Z}}(\varepsilon) \qquad |\varepsilon| = 1$$

$$\mathrm{Fun}(BS^1, \mathcal{D}(\mathbb{Z})) \simeq \mathrm{Mod}_{C_*(S^1)}$$

$$\tau_{\leq 1} \mathrm{Free}_{\mathbb{E}_1}(\varepsilon) \xrightarrow{\sim} C_*(S^1)$$

$$\tau_{\leq 1} \mathrm{Free}_{\mathbb{E}_1}(\varepsilon) \xrightarrow{\sim} A$$

lax monoidal

Thm $R \in \mathrm{Alg}(\mathcal{C})$.

HH refines to $\mathrm{Alg}(\mathcal{C}) \longrightarrow \mathrm{Fun}(BS^1, \mathcal{C})$.

which agrees w/ the S^1-action on $HH(R/\mathbb{Z})$ given by Connes operator.

paracyclic cat

Λ_∞ 1-Cat w/ $B\mathbb{Z}$-action:

- objects: totally ordered set w/ $\mathbb{Z}$-action equivalent to $\frac{1}{n}\mathbb{Z}$

- morphisms: $\mathrm{Hom}_{\Lambda_\infty}(\frac{1}{n}\mathbb{Z}, \frac{1}{m}\mathbb{Z})$

= order-preserving $\mathbb{Z}$-maps.

$B\mathbb{Z}$-action:

so $N\Lambda_\infty$ carries a $B\mathbb{Z} = S^1$-action.

* on objects ———

$\mathbb{Z}$ on morphisms: translation.

$$1\text{-Cat} \quad \Lambda^{\swarrow \text{cyclic cat}} := \Lambda_\infty / B\mathbb{Z}\text{-action}$$

<u>Lem</u> $\quad N\Lambda = (N\Lambda_\infty)_{hS^1}$

(as a result of some freeness)

<u>Lem</u> $\quad \mathrm{Fun}(N\Lambda, \mathcal{C}) \simeq \mathrm{Fun}(N\Lambda_\infty, \mathcal{C})^{S^1}$

trivial S^1-action

Indeed,

$\mathrm{Fun}(\mathrm{colim}\, A_\alpha, B) \simeq \lim \mathrm{Fun}(A_\alpha, B)$.

by applying $\mathrm{Map}(\mathcal{C}, -)$.

<u>Defn</u> A cyclic object in $\mathcal{C}$ is a functor $N(\Lambda^{op}) \longrightarrow \mathcal{C}$

have a functor

$$\Delta \longrightarrow \Lambda_\infty$$

$$[n-1] \longmapsto \mathbb{Z} \times [n-1]$$

$$\begin{gathered} \shortparallel \\ \cdots \amalg [n-1] \amalg [n-1] \amalg \cdots \\ \simeq \tfrac{1}{n}\mathbb{Z} \end{gathered}$$

Can define the underlying simplicial object of a cyclic object.

Lem The diagram

$$\begin{tikzcd} \mathrm{Fun}(\Lambda_\infty^{op}, \mathcal{C}) \arrow[r, "\mathrm{colim}"] \arrow[d, "\mathrm{res}"'] & \mathcal{C} \\ \mathrm{Fun}(\Delta^{op}, \mathcal{C}) \arrow[ur, "\mathrm{colim}"'] & \end{tikzcd}$$

commutes.

$\Delta^{op} \to \Lambda_\infty^{op}$ is cofinal.

is highly nontrivial however. see [NS17].

$\rightsquigarrow$ checked by using Quillen thm A.

?

Quillen A: For ∞-functor $F: \mathcal{C} \to \mathcal{D}$, if $\forall d \in \mathcal{D}$, $|N(F/d)|$

is contractible, that is, $\underline{F \text{ is homotopy cofinal}}$,

then (NF) is a weak equiv.

Lem X cyclic object.

then $\operatorname{colim}_{\Lambda_\infty^{op}} X \simeq \operatorname{colim}_{\Delta^{op}} X$ carrying

an S^1-action

Proof

$$\mathrm{Fun}(N\Lambda^{op}, \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(N\Lambda_\infty^{op}, \mathcal{C})^{hS^1} \xrightarrow{(\operatorname{colim})^{hS^1}} \mathcal{C}^{hS^1}$$

$$= \mathrm{Fun}(BS^1, \mathcal{C})$$

for $\mathcal{C}$ w/ trivial G-action, $\mathcal{C}^{hG} \simeq \mathrm{Fun}(BG, \mathcal{C})$.

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{D}, \mathcal{C}^{hG}) &\simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{D}, \mathcal{C})^{hG} \\
&\simeq \mathrm{Map}_{\mathrm{Kan}}(BG, \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{D}, \mathcal{C})) \\
&\simeq \mathrm{Map}_{\mathrm{Kan}}(BG, \mathrm{Fun}(\mathcal{D}, \mathcal{C})^{\simeq}) \\
&\simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(BG, \mathrm{Fun}(\mathcal{D}, \mathcal{C})) \\
&\simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{D}, \mathrm{Fun}(BG, \mathcal{C})).
\end{aligned}$$

Proof of Thm

i.e. HH refines to an S^1-object

$$HH(R/\mathcal{C}) = \operatorname{colim}\Big(\underbrace{\Delta^{op} \xrightarrow{\mathrm{cut}^{cyc}} \mathrm{Assoc}^{\otimes}_{act} \longrightarrow \mathcal{C}}\Big)$$

One can define $\Lambda_\infty^{op} \xrightarrow{\mathrm{cut}^{\mathbb{Z}}} \mathrm{Assoc}^{\otimes}_{act}$

$S \longmapsto$ set of $\mathbb{Z}$-equivariant cuts

that refines cut^{cyc}.

so $HH(R/\mathcal{C}) = \operatorname{colim}\big(\underline{\Lambda_\infty^{op} \to \mathrm{Assoc}^{\otimes}_{act}} \to \mathcal{C}\big)$

is $\mathbb{Z}$-invariant on morphisms, so factors through $\Lambda^{op} \to \mathrm{Assoc}^{\otimes}_{act}$

Rmk One can check Section B in [NS 17] as well.