The Tate Construction.

G - $\mathbb{E}_1$ - space.

We have $BG \in \mathcal{S}$.

$$\mathcal{C}^{BG} = \mathrm{Fun}(BG, \mathcal{C})$$

for any $\mathcal{C} \in \mathrm{Cat}_\infty$.

Construction, Consider G as an action on $\mathcal{S}^{B(G \times G)}$ by letting

$G \times G$ act on G as

$$(g, h) \cdot x = g x h^{-1}.$$

Def (Klein 2001)

define a spectrum $D_G \in Sp^{BG}$

called the <u>dualizing spectrum of G</u>

as $D_G = (\Sigma^\infty_+ G)^{h(G \times 1)}$

w/ its remaining $G = (1 \times G)$-action.

$BG = B(1 \times G)$

<u>Example</u>: G - finite group.

$$\Sigma^\infty_+ G = \bigoplus_{g \in G} \mathbb{S}$$

$$(\Sigma^\infty_+ G)^{hG} = \Big(\bigoplus_{g \in G} \mathbb{S}\Big)^{hG} \simeq \mathbb{S}.$$

Proof

$$H^*(G, \bigoplus_{g\in G} \pi_*(\mathbb{S})) \Rightarrow \pi_*((\bigoplus_{g\in G} \mathbb{S})^{hG})$$

$$H^*(G, \bigoplus_{g\in G} A) = \begin{cases} A, & * = 0 \\ 0, & * \neq 0 \end{cases}$$

$$\Rightarrow D_G \cong \mathbb{S}^{triv}.$$

If G is cpt Lie group,

Thm $D_G = \mathbb{S}^{\mathfrak{g}}$

and G acts on $\mathfrak{g}$ by Ad.

Example $T = U(1) = S^1$.

$$D_T = (\mathbb{S}^1)^{triv}.$$

Assume that

$$BG \simeq (M, m_0)$$

where M is a closed smooth mfld.

Thm We have $D_G := \mathbb{S}^{-T_{m_0}M}$

$$\left(\simeq \mathbb{S}^{-\dim M} \right)$$

as a functor

$$BG \simeq M \longrightarrow Sp$$

$$m \longmapsto \mathbb{S}^{-T_m M}$$

Rmk One can define for any space X, (in place of BG) a dualizing spectrum

$$D_X : X \longrightarrow Sp$$

e.g. $X \simeq \coprod BG_i$,

$$D_X|_{BG_i} = D_{G_i}.$$

Construction $\mathcal{C}$ stable ∞-cat w/ all limits & colimits.

For any $E \in Sp$, $X \in \mathcal{C}$, there is an object $E \otimes X \in \mathcal{C}$ (\$) ($\mathcal{C}$ is a module over Sp)

defined such that

$$- \otimes X : Sp \to \mathcal{C}$$

- sends colimits to colimits.
- $S \xrightarrow{\Sigma^\infty} Sp \xrightarrow{-\otimes X} \mathcal{C}$

is given by the functor

Const functor on X over M.

$$M \longmapsto M \otimes X = \operatorname{colim}_{M} c_X$$

<u>Def</u> $G \in Alg^{gp}_{\mathbb{E}_1}(S)$, $X \in \mathcal{C}^{BG}$.

We define the <u>norm map</u>

$$Nm_G : (DG \underset{S}{\otimes} X)_{hG} \longrightarrow X^{hG}$$

$\shortparallel$

denoted by Klein as X_{tG}

defined as the composite

X considered as $G\times G$-spectrum, where $G\times 1 \curvearrowright X$ trivially, $1\times G \curvearrowright X$ by G.

$$\Big((\Sigma^\infty_+ G)^{h(G\times 1)} \otimes_{\mathbb{S}} X \Big)_{h(1\times G)}$$

$$\downarrow$$

$$\Big(\big(\Sigma^\infty_+ G \otimes_{\mathbb{S}} X\big)^{h(G\times 1)} \Big)_{h(1\times G)}$$

$$\downarrow$$

$$\Big(\big(\Sigma^\infty_+ G \otimes_{\mathbb{S}} X\big)_{h(1\times G)} \Big)^{\underline{h(G\times 1)}}$$

$$\cong$$

$$X^{hG}$$

$\boxed{\text{Example}}$

(1) Suppose G finite.

$$X \in \mathcal{C}^{BG}.$$

$$\mathrm{Nm}_G : (D_G \otimes X)_{hG} \simeq X_{hG}$$

depends on choice of $D_G \simeq \mathbb{S}$

For $\mathcal{C} \in \mathrm{Sp}$, $X = HM$, $M \in \mathrm{Ab}^{BG}$

Then this is a map

$$\begin{tikzcd}
HM_{hG} \arrow[r, "\mathrm{Nm}_G"] \arrow[d] & HM^{hG} \\
H(M_G) \arrow[r] & H(M^G) \arrow[u]
\end{tikzcd}$$

this is induced from the classical

$$M_G \longrightarrow M^G, \quad [m] \longmapsto \sum_{g \in G} gm$$

This follows from the fact that for general X, the composit

$$X \longrightarrow X_{hG} \longrightarrow X^{hG} \longrightarrow X$$

is given by $\sum_{g \in G} \rho_G : X \to X$

(provided G is finite)

(2) For $G = \mathbb{T}$, we get

$$\Sigma X_{h\mathbb{T}} = (D_{\mathbb{T}} \otimes X)_{h\mathbb{T}} \longrightarrow X^{h\mathbb{T}}$$

$\boxed{\text{Thm}}$ The norm map

$$(D_G \otimes X)_{hG} \longrightarrow X^{hG}$$

is an equivalence provided that one of the following conditions holds:

① BG is finite CW cplx.

② X is induced, that is

$$X \simeq \Sigma^\infty_+ G \otimes Y,$$

where G acts on $\Sigma^\infty_+ G$.

Proof (1) ✓ (colimits / limits involved are finite) & filtered?

(2) — — -- are "close to finite".

<u>Example</u>. BG closed mfd M, $\mathcal{C} = \mathrm{Sp}$,

$$X = H\mathbb{Z}^{\mathrm{triv}}.$$

$$(DG \otimes X)_{hG} \xrightarrow{\sim} X^{hG} = H\mathbb{Z}^{BG}$$

$$\cong \qquad\qquad = \mathrm{map}(BG, X)$$

$$(\mathbb{S}^{-T_x M} \otimes H\mathbb{Z})_{hG} \qquad \pi_*(H\mathbb{Z}^{BG}) = H^{-*}(M, \mathbb{Z})$$

$$\cong$$

$$(H\mathbb{Z}[-n])_{hG}$$

needs verification.

$$\pi_* = H_{*+n}(M, \tilde{\mathbb{Z}})$$

local system for orientation cover

$$\rightsquigarrow$$

$$H_{*+n}(M, \tilde{\mathbb{Z}}) \xrightarrow{\sim} H^{-*}(M, \mathbb{Z})$$

Poincaré duality

- Replacing $H\mathbb{Z}$ by any spectrum we get Poincaré duality in arbitrary cohom theory.

- if $X = BG$ is finite CW cplx. the norm equivalence is a generalized Poincaré duality

$$H_*(X, D_X) \longrightarrow H^{-*}(X)$$

D_X is a parametrized sphere

($D_G = \mathbb{S}$[some degree] as underlying spectrum)

$\Longleftrightarrow$ X is Poincaré duality space

Thm (Klein, 2002)

The transformation

$$(DG \otimes -)_{hG} \longrightarrow (-)^{hG}$$

exhibits the functor $(DG \otimes -)_{hG}$ as the universal functor over $(-)^{hG}$ which preserves filtered colimits, ← I suppose

i.e., the assembly map.

Rmk e.g. when BG is finite, $(-)^{hG}$ already commutes w/ filtered colimits.

Rmk I guess (just some intuition) here should involve some $\mathfrak{G}$ function stuff.

Def For $X \in \mathcal{C}^{BG}$

(G is just a general $\mathbb{E}_1$-group)

the Tate construction is defined as

$$X^{tG} := \mathrm{cofib}\ \mathrm{Nm}_G$$

Example

① If BG is finite, then

$$X^{tG} = 0.$$

② For G finite and $X = HM$

w/ $M \in Ab^{BG}$

have

$$(HM)^{tG} = \mathrm{cofib}(\mathrm{Nm}_G : HM_{hG} \to HM^{hG})$$

$$\pi_*(-) = \hat{H}^{-*}(G, M)$$

Tate cohomology

Called Farrell-Tate cohomology for general G [John R. Klein, 2002]

Thm. $\mathcal{C}$ - s.m. ∞-cat s.t. $\otimes$ commutes w/ colimits in both variables separately.

Then $(-)^{tG} : \mathcal{C}^{BG} \to \mathcal{C}$ admits a (unique) lax s.m. struct. s.t.

$$(-)^{hG} \longrightarrow (-)^{tG}$$

admits a refinement to (lax) s.m. transformation

<u>Cor</u> If $A \in \mathrm{CAlg}(\mathcal{C}^{BG})$
$\simeq \mathrm{CAlg}(\mathcal{C})^{BG}$

then $A^{tG} \in \mathrm{CAlg}(\mathcal{C})$.

& $A^{hG} \longrightarrow A^{tG}$ is

a map of com. alg.