Tate diagonal

(in Spectra)

<u>Ob</u> For $A \in Ab$,

$$\begin{aligned} A &\longrightarrow A \otimes A \\ x &\longmapsto x \otimes x \end{aligned}$$

is not a homo. (w/ error term

$\underbrace{x \otimes y + y \otimes x}$). $= N(x \otimes y)$

What if we quotient the error term?

$$N : (A^{\otimes p})_{C_p} \longrightarrow (A^{\otimes p})^{C_p}$$

$$[x_1 \otimes \cdots \otimes x_p] \longmapsto \sum_{\sigma \in C_p} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(p)}$$

The "diagonal" $A \longrightarrow A^{\otimes p}$

induces a homomorphism

$$\Delta: A \longrightarrow (A^{\otimes p})^{C_p} / N((A^{\otimes p})_{C_p})$$

which factors thru an iso

$$A/p \xrightarrow{\sim} (A^{\otimes p})^{C_p} / N((A^{\otimes p})_{C_p})$$

<u>Thm</u> There is a unique lax s.m. nat. transformation

$$\mathrm{id}_{Sp} \Longrightarrow ((-)^{\otimes p})^{tC_p}.$$

called top Singer construction

Lem. $X \mapsto (X^{\otimes p})^{tC_p}$ is exact.

used Day convolution of filtered spectra

Rmk $((\Sigma X)^{\otimes p})^{tC_p} \simeq \Sigma (X^{\otimes p})^{tC_p}$

Indeed, $(\Sigma X)^{\otimes p} \simeq \mathbb{S}^p \otimes X^{\otimes p}$

but C_p-action acts nontrivially on $\mathbb{S}^p$.

(indeed the regular rep sphere).

$$(\mathbb{S}^p)^{tC_p} = \mathbb{S}^1_{\text{triv.}}$$

Lem ("stable Yoneda")

$\mathcal{C}$ - stable ∞-cat.

$\mathrm{Fun}^{ex}(\mathcal{C}, Sp)$ is stable

$$\operatorname{map}_{\mathrm{Fun}^{ex}(\mathcal{C}, \mathrm{Sp})}(\operatorname{map}(X, -), F)$$

$$\simeq F(X)$$

nat. equiv.

<u>Cor</u> Put $X = \mathbb{S}$,

Nat. transformation

$$X \longrightarrow (X^{\otimes p})^{tC_p}$$

corresponds to $\mathbb{S} \longrightarrow (\mathbb{S}^{\otimes p})^{tC_p} \simeq \mathbb{S}^{tC_p}$

$$\mathbb{S} \longrightarrow \mathbb{S}^{hC_p} \xrightarrow{\mathrm{can}} \mathbb{S}^{tC_p}$$

$$\begin{gathered}\mathbb{S}^{hC_p} \\ \wr\vert \\ \operatorname{map}(\Sigma^\infty_+ BC_p, \mathbb{S})\end{gathered}$$

[btw. indeed $\mathbb{S}^{tC_p} \simeq \mathbb{S}^\wedge_p$ by Sullivan Conjecture)

Lem

map$(\mathbb{S}, -)$ is initial among lax s.m. exact functors $Sp \to Sp$.

Rmk Day convolution in $\mathrm{Fun}^{ex}(Sp, Sp)$ struct.

Thm For X bounded below spectrum, then

$$X \longrightarrow (X^{\otimes p})^{tC_p}$$

exhibits $(X^{\otimes p})^{tC_p}$ as the p-completion of X.

In particular, $(X^{\otimes p})^{tC_p}$ is again bounded below.

special form of the Segal conj.
proved by Carlson

An interesting stuff:

inspect the spectral sequence for computing $\pi_*(X^{tC_p})$.

knows much more about R/p

Defn For $\mathbb{E}_\infty$-ring R, have <u>Tate-valued Frobenius</u>

$$R \xrightarrow{\Delta} (R^{\otimes p})^{tC_p} \xrightarrow{\mu^{tC_p}} R^{tC_p}$$

(used that $\mu: R^{\otimes p} \to R$ is

eqvt wrt C_p-action,
same as $(R^{\otimes p})_{hC_p} \longrightarrow R$)

(Spectral analogue of $R \xrightarrow{\varphi} R/p$
$x \longmapsto x^p$
for ordinary com. rings)

Applied to map $(\Sigma^\infty_+ X, E)$,
(E some com. ring spectrum)
get <u>Power operations</u>.