TP

Defn $TP(R) = THH(R)^{tS^1}$.

Since $(-)^{tG}$ is lax s.m. in a way compatible w/ $(-)^{hG}$, so $TP(R)$ is a $TC^-(R)$-alg if R is $\mathbb{E}_\infty$.

Prop For $X \in Sp^{BG}$, there is a multi. conditionally convergent SS

$$\pi_*\left((H\pi_*(X))^{tG} \right) \Rightarrow \pi_*(X^{tG})$$

just consider the Whitehead $\tau_{\geq \bullet} X$ applying $(-)^{tG}$.

somewhat delicate, it does not commute w/ colimit/limit

For $H\mathbb{Z}$, $G = S^1$, consider the cofiber sequence

$$(S^{\mathrm{ad}G} \otimes H\mathbb{Z})_{hG} \xrightarrow{\mathrm{Nm}} H\mathbb{Z}^{hS^1} \longrightarrow H\mathbb{Z}^{tS^1}$$

$$(S^{\mathrm{ad}G} \otimes H\mathbb{Z})_{hG} \simeq \Sigma H\mathbb{Z}_{hS^1}$$

actually doesn't do anything. $H\mathbb{Z}^{hS^1}$ has htpy groups in nonpositive deg. while $\Sigma H\mathbb{Z}_{hS^1}$ in nonnegative deg

<u>Ex</u> Compute $\pi_*(H\mathbb{Z}^{tS^1})$ additively

<u>A</u>:
$$\pi_* H\mathbb{Z}^{tS^1} = \begin{cases} \mathbb{Z}, & \text{for } * \text{ even} \\ 0, & * \text{ odd.} \end{cases}$$

$\pi_* H\mathbb{Z}^{hS^1} \cong \mathbb{Z}[t]$ $\quad |t| = -2$

$\pi_* \Sigma H\mathbb{Z}_{hS^1} \cong \mathbb{Z}[s][1]$, $|s| = 2$

$\mathbb{Z}$	$\to 0$	$\to ?$		3
0	$\to 0$	$\to ?$		2
$\mathbb{Z}$	$\to 0$	$\to ?$		1
0	$\to \mathbb{Z}$	$\to ?$		0
0	$\to 0$	$\to ?$		-1
0	$\to \mathbb{Z}$	$\to ?$		-2
0				

Lem $\pi_* H\mathbb{Z}^{tS^1} \cong \mathbb{Z}[t^{\pm}]$

(as $\pi_* H\mathbb{Z}^{hS^1}$-alg)

Proof Consider the Tate SS for $H\mathbb{Z} \otimes \underline{\Sigma^\infty_+ S^1}$

the free S^1-module

$(-)\otimes \Sigma^\infty_+ S^1$: induction

We know $(H\mathbb{Z} \otimes \Sigma^\infty_+ S^1)^{tS^1} = 0$

free action ?

See [NS] Lem I.3.8

or Nota Tate (p11) Thm

$(2i, j)$

$\cdot\ x_i \sigma_j$

$$\begin{array}{ccccccccc}
\cdots & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 \\
\cdots & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 \ \cdots
\end{array}$$

these "$\nwarrow$" are iso.

Tate SS $(H\mathbb{Z})$ acts on Tate SS$(H\mathbb{Z} \otimes \Sigma^\infty_+ S^1)$

Fix gen. $x_{-i} = t^i$.

σ_0, σ_1 of $H_*(S^1)$

$d_2(\sigma_0) = \pm t \cdot \sigma_1$,

So $\quad d_2(x_i \cdot \sigma_0) = \pm x_i \cdot t\sigma_1$

using $H\mathbb{Z}$ acts on $H\mathbb{Z} \otimes \Sigma^\infty_+ S^1$.

$$\Rightarrow \quad \pm x_{i-1} \sigma_1 = \pm x_i \cdot t\sigma_1$$

$$\Rightarrow \quad x_i \cdot t = \pm x_{i-1}.$$

Make a good choice of x_i

$$\rightarrow \quad \pi_* H\mathbb{Z}^{tS^1} = \mathbb{Z}[t^\pm].$$

<u>Ex</u> Similarly, compute $\pi_* H\mathbb{Z}^{tC_m}$ as a ring. (using a free action of C_m on a space w/ easy homology groups)

$$\pi_*(H\mathbb{Z}^{hC_m}) = \begin{cases} \mathbb{Z}, & * = 0 \\ \mathbb{Z}/m\mathbb{Z}, & * = 2k < 0 \\ 0, & \text{else} \end{cases}$$

$$\pi_*(H\mathbb{Z}_{hC_m}) = \begin{cases} \mathbb{Z}, & * = 0, \\ \mathbb{Z}/m\mathbb{Z}, & * = 2k+1 > 0 \\ 0, & \text{else} \end{cases}$$

$$\pi_*(H\mathbb{Z}^{tC_m}) = \begin{cases} \mathbb{Z}/m\mathbb{Z}, & * \text{ even} \\ 0, & * \text{ odd} \end{cases}$$

Work analogously for $\underset{C_m}{\circlearrowleft}\Sigma^\infty_+ S^1 \otimes_{\mathbb{S}} H\mathbb{Z}$

get $\quad \pi_*(H\mathbb{Z}^{tC_m}) = C_m[t^\pm] \quad |t| = -2$

S^1 looks like some "$\mathbb{Z}$ version" of C_m.

Rmk Same argument w/ $HA \otimes \Sigma^\infty_+ S^1$ as $H\mathbb{Z}$-module get $\pi_* HA^{tS^1} = A[t^{\pm 1}]$

Prop If X is an object of $\mathrm{Fun}(BS^1, \mathrm{Mod}_{H\mathbb{Z}})$ (equivalently, a module over $H\mathbb{Z}^{triv}$ in $\mathrm{Fun}(BS^1, Sp)$)

then $X^{tS^1} \simeq X^{hS^1} \otimes_{H\mathbb{Z}^{hS^1}} H\mathbb{Z}^{tS^1}$

$$\pi_* X^{tS^1} \simeq \pi_* X^{hS^1}[t^{-1}]$$

Sketch

$$H\mathbb{Z}^{tS^1} \simeq \mathrm{Colim}\left(H\mathbb{Z}^{hS^1} \xrightarrow{t} \Sigma^2 H\mathbb{Z}^{hS^1} \xrightarrow{t} \cdots\right)$$

as $H\mathbb{Z}^{hS^1}$ – module.

need to show

$$X^{tS^1} = \mathrm{colim}\left(X^{hS^1} \xrightarrow{t} \Sigma^2 X^{tS^1} \xrightarrow{t} \cdots \right)$$

proves for X having finitely many $\pi_* \neq 0$

- works for $X \in M$.
- Both sides compatible w/ fiber seq. ← using Postnikov tower
- to extend to all X, need some connectivity argument

Prop $HP(R) = HH(R)^{tS^1}$

follows from the previous prop.

Prop Tate s.s. for S^1 takes the form

$$\pi_*(X)[t^{\pm}] \Rightarrow \pi_*(X^{tS^1})$$

$$THH_*(R)[t^{\pm}] \Rightarrow TP_*(R)$$

Thm $\pi_* TP(\mathbb{F}_p) = \mathbb{Z}_p[t^{\pm 1}]$

cannot apply previous prop since $TP(\mathbb{F}_p)$ as $H\mathbb{Z}$-module needn't be S^1-equivariant.

Proof Consider Tate S.S.

(it's a periodic version of the HFPSS)

...

$$\pi_{-2k} TP(\mathbb{F}_p) = \pi_{-2k} TC^-(\mathbb{F}_p)$$
$$= \mathbb{Z}_p \cdot (\tilde{t})^k.$$

Can also choose a rep $\widetilde{t^{-1}}$ in $\pi_2 TP$.

have $\tilde{t} \cdot \widetilde{t^{-1}} = 1$ (as well as in higher fil.)

$\Rightarrow \tilde{t} \cdot \widetilde{t^{-1}} = \text{unit}$

$\Rightarrow \tilde{t}$ invertible, and

$\tilde{t}^{-k}$ generates $\pi_{2k} TP$.

Done.

in $\mathbb{E}_2$ page of the SS, one can notice that the diagonal indeed suggests the canonical fil. of $\mathbb{Z}_p$.

<u>Warning</u>: $TP(R)$ needn't be periodic, since that previous <u>prop</u> doesn't hold at a free cost, we're indeed lucky here.