THH of $\mathbb{Z}$

Thm (Bökstedt)

$$THH_*(\mathbb{Z}) = \begin{cases} \mathbb{Z}, & * = 0 \\ \mathbb{Z}/n, & * = 2n-1 \\ 0, & o/w \end{cases}$$

In fact $THH_*(\mathbb{Z})$ is the homology of the DGA

$\mathbb{Z}[x] \otimes \Lambda(e)$, $|x| = 2$, $|e| = 1$, $\partial x = e$, $\partial e = 0$.

Rmk In fact $THH(\mathbb{Z})$ is given by this DGA as an $\mathbb{E}_1$-algebra over $H\mathbb{Z}$.

As a consequence,

$$THH(\mathbb{Z})/p = THH(\mathbb{Z}) \otimes_{H\mathbb{Z}} H\mathbb{F}_p$$

is iso. in homotopy groups to the homology of DGA $(\mathbb{F}_p[x] \otimes \Lambda(e), \partial)$

Defn For $R \in \mathrm{Alg}(\mathrm{Sp})$, define

$$\mathrm{THH}(R; \mathbb{Z}_p) := \mathrm{THH}(R)^{\wedge}_p$$

$$\mathrm{THH}(R; \mathbb{Q}) := \mathrm{THH}(R)_{\mathbb{Q}}$$

$$\mathrm{THH}(R; \mathbb{Q}_p) := \mathrm{THH}(R, \mathbb{Z}_p)_{\mathbb{Q}}$$

Recall

$$\begin{tikzcd}
\mathrm{THH}(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \prod_p \mathrm{THH}(R; \mathbb{Z}_p) \arrow[d] \\
\mathrm{THH}(R; \mathbb{Q}) \arrow[r] & \underline{\mathbb{Q} \otimes \prod_p \mathrm{THH}(R; \mathbb{Z}_p)}
\end{tikzcd}$$

Lem

$$\mathrm{THH}(R; \mathbb{Z}_p) = \mathrm{THH}(R^{\wedge}_p; \mathbb{Z}_p)$$
$$= \mathrm{HH}(R^{\wedge}_p / \mathrm{Sp}^{\wedge}_p)$$

$$\mathrm{THH}(R;\mathbb{Q}) = \mathrm{THH}(R_{\mathbb{Q}}/\mathbb{Q})$$
$$= \mathrm{HH}(R_{\mathbb{Q}}/\mathbb{Q})$$

$\mathrm{Sp}_{\mathbb{Q}} \simeq \mathrm{Mod}_{H\mathbb{Q}}$
$\simeq D(\mathbb{Q})$

Cor $\mathrm{THH}(\mathbb{Z};\mathbb{Q}) = H\mathbb{Q}$

Thm
$$\mathrm{THH}_*(\mathbb{Z};\mathbb{Z}_p) = \begin{cases} \mathbb{Z}_p & * = 0 \\ \mathbb{Z}_p/n\mathbb{Z}_p, & * = 2n-1 \\ 0, & \text{o/w} \end{cases}$$
$$\simeq H_*\left(\mathbb{Z}_p[x] \otimes \Lambda(e), \partial\right)$$

This thm gives Bökstedt's thm.

Defn We define an E_∞-ring
$$\mathbb{S}[\mathbb{Z}] = \mathbb{S}[\mathbb{N}] := \Sigma^\infty_+ \mathbb{N}.$$

viewed as a com. algebra in $(\mathcal{S}, \times)$.

To give a discrete com. ring R the struct. of an $\mathbb{S}[z]$-alg is equiv. to give an element $\pi \in R$,

$z \mapsto \pi$:

$$\begin{tikzcd} \mathbb{S}[z] \arrow[r] \arrow[dr] & H\mathbb{Z}[z] \arrow[d] \\ & HR \end{tikzcd}$$

Consider $\mathbb{Z}$ as an $\mathbb{S}[z]$-alg by sending $z \mapsto p$,

<u>Warning</u> if considering instead an $\mathbb{E}_\infty$-ring R, then an $\mathbb{E}_\infty$-map $\mathbb{S}[z] \to R$ is not the same as an element $z \in \pi_0 R$, in other words, $\mathbb{S}[z]$ is not the

free $\mathbb{E}_\infty$-alg on a single generator.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad F$

($\mathbb{S}[z]$ is free as an $\mathbb{E}_1$-alg though)

<u>Thm</u> (rel. Bökstedt periodicity)

We have $\mathrm{THH}_*(\mathbb{Z}/\mathbb{S}[z]\,;\mathbb{Z}_p) \simeq \mathbb{Z}_p[x]$,

$|x| = 2$.

remember that $\mathbb{Z}$ is an $\mathbb{S}[z]$-algebra

<u>Proof</u> $\quad \mathrm{THH}(\mathbb{Z}/\mathbb{S}[z]\,;\mathbb{Z}_p)/p$

$$\simeq \mathrm{THH}(\mathbb{Z}/\mathbb{S}[z]\,;\mathbb{Z}_p) \otimes_{H\mathbb{Z}} H\mathbb{F}_p$$

need condition $H\mathbb{Z}$-module

$$\simeq \mathrm{THH}(\mathbb{Z}/\mathbb{S}[z]\,;\mathbb{Z}_p) \otimes_{\mathbb{S}[z]} \mathbb{S}$$

$$\simeq \mathrm{THH}(\mathbb{Z} \otimes_{\mathbb{S}[z]} \mathbb{S} / \mathbb{S}; \mathbb{Z}_p)$$

$$\simeq \mathrm{THH}(\mathbb{F}_p; \mathbb{Z}_p)$$

$$\simeq \mathrm{THH}(\mathbb{F}_p)$$

consider LES for

$$\mathrm{THH} \xrightarrow{p} \mathrm{THH} \longrightarrow \mathrm{THH}/p$$

$\Rightarrow$ $\mathrm{THH}_*(\mathbb{Z}/\mathbb{S}[z]; \mathbb{Z}_p)$ is concentrated

in even deg & is p-torsion free

$\Rightarrow$ $\exists\, x \in \mathrm{THH}_2(\mathbb{Z}/\mathbb{S}[z]; \mathbb{Z}_p)$

lifting $x \in \mathrm{THH}_2(\mathbb{F}_p)$.

get $\mathbb{Z}_p[x] \longrightarrow \mathrm{THH}_*(\mathbb{Z}/\mathbb{S}[z]; \mathbb{Z}_p)$

it's an iso (by passing to p-completion).

$$\mathrm{THH}(R/\mathbb{S}[z]) \overset{\text{base change}}{=} \mathrm{THH}(R) \otimes_{\mathrm{THH}(\mathbb{S}[z])} \mathbb{S}[z]$$

$$= \mathrm{THH}(R) \otimes_{H\mathbb{Z} \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{S}[z])} H\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{S}[z]$$

$$\overset{\text{base change}}{=} \mathrm{THH}(R) \otimes_{\mathrm{HH}(\mathbb{Z}[z]/\mathbb{Z})} \mathbb{Z}[z]$$

HKR fil.

Prop have a convergent, multiplicative spectral sequence

$$\mathrm{THH}_n(R/\mathbb{S}[z]) \otimes_{\mathbb{Z}[z]} \Omega^m_{\mathbb{Z}[z]/\mathbb{Z}} \Rightarrow \mathrm{THH}_{n+m}(R)$$

$$\cong$$

$$\mathrm{THH}_*(R/\mathbb{S}[z]) \otimes_{\mathbb{Z}} \Lambda(dz)$$

So $\mathbb{Z}_p[x] \otimes_{\mathbb{Z}_p} \Lambda(dz) \Rightarrow THH_*(\mathbb{Z}; \mathbb{Z}_p)$