The Definition of TC

Recall we have a $\mathbb{T}$-eqvt map

$$THH(R) \longrightarrow THH(R)^{tC_p}$$

Defn The ∞-cat of cyclotomic spectra is define as the pullback

$$\begin{tikzcd}
\mathrm{CycSp} \arrow[r] \arrow[d] & \prod_p (Sp^{B\mathbb{T}})^{\Delta^1} \arrow[d] \\
Sp^{B\mathbb{T}} \arrow[r, "(id,\, (-)^{tC_p})"] & \prod_p (Sp^{B\mathbb{T}} \times Sp^{B\mathbb{T}})
\end{tikzcd}$$

In particular, a map of cyclotomic spectra (X, φ_p) to (Y, φ_p') is given by

(1) $\mathbb{T}$-eqvt map $X \xrightarrow{f} Y$

(2) $\mathbb{T}$-eqvt hpty

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi_p"'] & Y \arrow[d, "\varphi_p'"] \arrow[dl, phantom, "\swarrow"] \\ X^{tC_p} \arrow[r, "f^{tC_p}"'] & Y^{tC_p} \end{tikzcd}$$

The mapping space is given as the equalizer

$$\mathrm{Map}_{\mathrm{CycSp}}(X,Y) \longrightarrow \mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}(X,Y) \underset{(\varphi_p')_*}{\overset{\varphi_p^*}{\rightrightarrows}} \mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}(X, Y^{tC_p})$$

<u>Prop</u> (1) CycSp is stable

and we have a similar recognition of $\mathrm{map}_{\mathrm{CycSp}}$.

(2) CycSp has all limits & colimits

(it's indeed presentable)

(3) CycSp has a s.m. struct.

given by

$$(X, \varphi_p) \otimes (Y, \varphi_p') := \left(X \underset{\mathbb{S}}{\otimes} Y, \begin{array}{c} X \otimes Y \\ \downarrow \varphi_p \otimes \varphi_p' \\ X^{tC_p} \otimes Y^{tC_p} \\ \downarrow \\ (X \otimes Y)^{tC_p} \end{array} \right),$$

(4) A com. alg. in CycSp is given by :

(i) $X \in \mathrm{CAlg}(\mathrm{Sp})^{B\mathbb{T}}$

(ii) a $\mathbb{T}$-eqvt map of com. alg

$$\varphi_p : X \longrightarrow X^{tC_p} \quad \forall p$$

In particular, THH(R) for R $\mathbb{E}_\infty$ is an com. alg. in CycSp.

<u>Recall</u>

$$TC^-(R) := THH(R)^{hS^1}$$
$$= \mathrm{map}_{Sp^{BT}}(\mathbb{S}^{triv}, THH(R))$$

We give similarly description for TC.

it's not an analogue of HC but a refinement of TC^-.

Example: $\mathbb{S}^{triv}$ canonically a CycSp

- underlying is $\mathbb{S}^{triv}$
- $\varphi_p : \mathbb{S} \to \mathbb{S}^{tC_p}$ is

the unit, i.e.

$$\mathbb{S} \longrightarrow \underset{\substack{\mathrel{\rotatebox{90}{$\subseteq$}} \\ \mathrm{map}(\Sigma^\infty_+ BC_p, \mathbb{S})}}{\mathbb{S}^{hC_p}} \xrightarrow{\text{can}} \mathbb{S}^{tC_p}$$

Ex Lift

$$\mathbb{S} \to \mathbb{S}^{hC_p} \longrightarrow \mathbb{S}^{tC_p}$$

to a diagram of $\mathbb{T}$-eqvt maps.

suffices to lift it to $\mathbb{S} \to (\mathbb{S}^{tC_p})^{h\mathbb{T}/C_p}$

$$\mathbb{S} \to \mathbb{S}^{h\mathbb{T}} \simeq (\mathbb{S}^{hC_p})^{h\mathbb{T}/C_p} \longrightarrow (\mathbb{S}^{tC_p})^{h\mathbb{T}/C_p}$$

Indeed, this $\mathbb{S}^{triv} \in$ CycSp coincide w/ $THH(\mathbb{S}) \in$ CycSp.

<u>Defn</u> For $R \in \mathrm{Alg}(\mathrm{Sp})$,

$$TC(R) := \mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, THH(R))$$

- If R is $\mathbb{E}_\infty$, then $TC(R)$ is $\mathbb{E}_\infty$-ring spectrum.

- More generally, for $X \in \mathrm{CycSp}$, can define

$$TC(X) := \mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, X)$$

$$TC(R) = TC(THH(R))$$

A quick calculation gives

$$TC(X) \simeq Eq\left(X^{h\mathbb{T}} \rightrightarrows \prod_p (X^{tC_p})^{h\mathbb{T}}\right)$$

$$\simeq fib\left(X^{h\mathbb{T}} \xrightarrow{\varphi_p^{h\mathbb{T}} - can} \prod_p (X^{tC_p})^{h\mathbb{T}}\right)$$

where,

$$can: X^{h\mathbb{T}} = (X^{hC_p})^{h\mathbb{T}/C_p} \xrightarrow{can^{h\mathbb{T}/C_p}} (X^{tC_p})^{h\mathbb{T}/C_p} = (X^{tC_p})^{h\mathbb{T}}$$

Thm [NS] Lem II.4.2 Assume X is bounded below.

Then

$$X^{t\mathbb{T}} \to (X^{tC_p})^{h\mathbb{T}}$$

exhibits $(X^{tC_p})^{h\mathbb{T}}$ as the p-completion of $X^{t\mathbb{T}}$.

Moreover, if X is p-complete, then so is

$X^{t\mathbb{T}}$.

As a result, if $X \in \mathrm{CycSp}$, w/ underlying spectrum bounded below,

then $\varphi_p^{h\mathbb{T}}: X^{h\mathbb{T}} \longrightarrow (X^{tC_p})^{h\mathbb{T}} = (X^{t\mathbb{T}})^\wedge_p$

$$\varphi: X^{h\mathbb{T}} \longrightarrow (X^{t\mathbb{T}})^\wedge := \prod_p (X^{t\mathbb{T}})^\wedge_p$$

Cor for such X

$$TC(X) \simeq \mathrm{fib}\left(X^{h\mathbb{T}} \xrightarrow{\varphi - \mathrm{can}} (X^{t\mathbb{T}})^\wedge\right)$$

$$= \mathrm{Eq}\left(X^{h\mathbb{T}} \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} (X^{t\mathbb{T}})^\wedge\right)$$

If R is connective

$$TC(R) \simeq \mathrm{Eq}\left(TC^-(R) \underset{\mathrm{can}}{\overset{\varphi}{\rightrightarrows}} TP(R)^\wedge\right)$$

For X a qcqs scheme,

define

$$\mathrm{THH}(X) := \lim_{\substack{U \subset X \\ \text{affine open}}} \mathrm{THH}(\mathcal{O}_U)$$

$$\mathrm{TC}^-(X) := \mathrm{THH}(X)^{hT} = \lim_{\substack{U \subset X \\ \text{affine open}}} \mathrm{TC}^-(\mathcal{O}_U)$$

$$\mathrm{TP}(X) := \mathrm{THH}(X)^{tT} = \lim_{\substack{U \subset X \\ \text{affine open}}} \mathrm{TP}(\mathcal{O}_U)$$

$$TC(X) := TC(THH(X))$$

$$\simeq \lim_{U \subset X} TC(\mathcal{O}_U)$$

$$\simeq \mathrm{Eq}\,(TC^-(X) \rightrightarrows TP(X)^\wedge)$$

<u>Rmk</u> One can show for $X \in \mathrm{Cycsp}$

n-connective that

$$TC(X)^\wedge_p = TC(X; \mathbb{Z}_p)$$

is $(n-1)$-connective.

In particular,

$TC(R; \mathbb{Z}_p)$ for R ring

(or connective ring spectrum)

is (-1)-connective.