TC of Perfect Rings

Recall $\quad TC_*(\mathbb{F}_p) = \begin{cases} \mathbb{Z}_p, & * = 0, -1 \\ 0, & o/w. \end{cases}$

k perfect $\mathbb{F}_p$-alg.

like $\mathbb{F}_{p^n}$, $\prod k_i$ w/ k_i perfect p-alg

$C^0(X, \mathbb{F}_p)$, $X \in \mathrm{Top}$

$\mathbb{F}_p[x^{\mathbb{Z}/p^\infty}]$

We have $\quad THH_*(k) = k[x]$

(generalised Bökstedt)

Recollection on Witt Vectors

There exist com. rings $W(k)$ called p-typical Witt vectors of k w/ the following property:

(1) $W(k)$ is p torsion-free

(2) $W(k)$ is p-complete

(3) $W(k)/p \cong k$.

Moreover, $W(k)$ is uniquely determined by 1, 2, 3.

recall that it's defined using cotangent complex & obstruction theory.

Thm R - derived p-complete ring

i.e. HR is p-complete

or R is p-complete in $\mathcal{D}(\mathbb{Z})$

Then $\mathrm{Hom}_{\mathrm{CRing}}(W(k), R) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CRing}}(k, R/p)$

Proof If R is p-complete, $R = \lim R/p^n$.

Inductively lift along $R/p^{n+1} \to R/p^n$.

If R is derived p-complete,

$R = \lim R/\!/p^n$.

Reduce to $R/\!/p$, then use

$R/\!/p \to R/p$ zero extension.

Cor We have equivalence of cats

$$\left\{ \begin{array}{c} p\text{-torsion free, } p\text{-complete} \\ \text{rings w/ perfect mod } p \text{ reduction} \end{array} \right\} \underset{W}{\overset{/p}{\underset{\sim}{\rightleftarrows}}} \left\{ \begin{array}{c} \text{perfect} \\ \mathbb{F}_p\text{-algs} \end{array} \right\}$$

Cor We have iso of rings:

$$F : W(k) \longrightarrow W(k)$$

called <u>Witt vector Frobenius</u>

s.t. $F/p : k \to k$ is the Frobenius

$$\varphi : x \longmapsto x^p.$$

Prop k perfect $\mathbb{F}_p$-alg.

Then we have natural isos

(in k)

$$TC^-(k) \cong W(k)[x,t]/xt-p$$

$$TP(k) \cong W(k)[t^{\pm}]$$

Sketch Study HFPSS for $THH(k)^{hT}$

and SS-map for $k \to \mathbb{F}_p$.

Now, have

$$\mathrm{can}, \varphi : TC^-(k) \longrightarrow TP(k)$$

- $\mathrm{can}(x) = p t^{-1}$, $\mathrm{can}(t) = t$, $\mathrm{can}(r) = r$, $r \in W(k)$
- $\varphi(x) = t^{-1}$, $\varphi(t) = pt$

Rmk For any connective ring spectrum, have iso $\pi_0 \mathrm{can} : TC_0^-(R) \xrightarrow{\sim} TP_0(R)$.

Thus $\pi_0 \varphi$ can be viewed as an endo of $TC_0^-(R) = TP_0(R)$

Prop $\pi_0 \varphi : W(k) \to W(k)$ is the Witt vector Frobenius F.

Proof

We have

$$TP(k)^\wedge = TP(k)$$

$$\begin{tikzcd}
 & TC^-(k) \arrow[r, "\varphi"] \arrow[d] & TP(k) \arrow[d] & \\
k \arrow[r] \arrow[rrr, bend right, "\mathbb{E}_\infty \text{ Frobenius}"'] & THH(k) \arrow[r, "\varphi_p"] & THH(k)^{tC_p} \arrow[r] & k^{tC_p}
\end{tikzcd}$$

Apply π_0 :

$$\begin{tikzcd}
 & W(k) \arrow[r, "\pi_0\varphi"] \arrow[d, "/p"] & W(k) \arrow[d] \arrow[dr, "p"] & \\
k \arrow[r, "id"] & k \arrow[r, "\pi_0\varphi_p"] & ? \arrow[r] & k
\end{tikzcd}$$

don't know why

$\pi_0\varphi :$ $W(k) \longrightarrow W(k)$ is F

by the previous equiv of cats.

Cor

$$TP_*(k) = \begin{cases} W(k)^F & , * = 0 \\ W(k)_F & , * = -1 \\ 0 , & o/w \end{cases}$$

$W(k)^F = \ker(\mathrm{id} - F)$,

$W(k)_F = \mathrm{coker}(\mathrm{id} - F)$.

Facts: ① $W(k)^F \simeq W(k^\varphi)$

Cor For k field, $TC_0(k) = \mathbb{Z}_p$.

② In general,

$k^\varphi = C^0(X, \mathbb{F}_p)$ called Stone duality

where X is a profinite space given

by $\mathrm{Spec}(k^\varphi)$

also the space of connected components of $\mathrm{Spec}(k)$

③ $W(k^{\flat}) = W(C^0(X, \mathbb{F}_p))$
$$= C^0(X, \mathbb{Z}_p)$$
$$\cong C^0(\mathrm{Spec}(k^{\flat}), \mathbb{Z}_p)$$

<u>Cor</u> Fix k a perfect $\mathbb{F}_p$-alg.

then $TC_0(k) = C^0(\mathrm{Spec}(k), \mathbb{Z}_p)$
$$= C^0(\mathrm{Spec}(k^{\flat}), \mathbb{Z}_p)$$

④ $TC_{-1}(k) = W(k)_F$ is p-torsion free and its mod p reduction is k_F.

Thus $TC_{-1}(k)$ is p-completely free of rank the dim of k_φ, i.e.,

$$TC_{-1}(k) = \left(\bigoplus_S \mathbb{Z}_p\right)^\wedge_p$$

for some index set S w/

$$k_\varphi = \bigoplus_S \mathbb{F}_p$$

<u>Cor</u> If $k_\varphi = 0$, e.g. $k = \bar{k}$

then $TC_{-1}(k) = 0$

<u>Example</u> $TC_*(\mathbb{F}_p) = H\mathbb{Z}_p$

a general fact: for any ring k and any ring homo $\varphi: k \to k$, k_φ is a k^φ-mod

⑤ Say $k^\varphi = \underbrace{\mathbb{F}_p \times \cdots \times \mathbb{F}_p}_{r}$.

Then k_φ is a k^φ-module

can be described as

$$k_\varphi = V_1 \times \cdots \times V_r \quad \text{w/} \quad V_j \in \mathrm{Vect}_{\mathbb{F}_p}$$

$$\Rightarrow W(k)_F = \tilde{V}_1 \times \cdots \times \tilde{V}_r$$

as a $W(k)^F = \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{r}$ - module,

where $\tilde{V}_i$ is a p-complete, p-torsion free lift of V_i to $\mathbb{Z}_p$.