$TC(\mathbb{F}_p)$

Recall

$$TC(R) = \mathrm{Eq}\left(TC^-(R) \underset{can}{\overset{\varphi}{\rightrightarrows}} TP(R)^\wedge \right)$$

for R discrete, bounded below

$$\mathrm{can}: TC^-(R) \xrightarrow{can} TP(R) \longrightarrow TP(R)^\wedge$$

$$\varphi: TC^-(R) \xrightarrow{(\varphi_p^{hT})_p} \prod_p \left(THH(R)^{tC_p}\right)^{hT}$$

$$= \prod_p \left(THH(R)^{tT}\right)^\wedge_p$$

$$=: TP(R)^\wedge$$

(don't know why Thomas mentioned profinite completion of TP here)

looks like $(-)^\wedge$ has another meaning:

for G a discrete group. $\hat{G} := \lim^{\mathrm{TorGrp}}_{N \trianglelefteq G,\ [G:N] < \infty} G/N$

e.g. $\hat{\mathbb{Z}} = \lim_n \mathbb{Z}/n\mathbb{Z} \cong \prod_p \mathbb{Z}_p$

<u>Recall</u> as proved before,

- $TC_*^-(\mathbb{F}_p) = \mathbb{Z}_p[x,t]/xt-p$,

$$|x| = 2, \ |t| = -2$$

- $TP_*(\mathbb{F}_p) = TC_*^-(\mathbb{F}_p)[t^{-1}]$

$$= \mathbb{Z}_p[t^{\pm}]$$

$\Rightarrow TP(\mathbb{F}_p) = TP(\mathbb{F}_p)^\wedge$.

- can, $\varphi : TC_*^-(\mathbb{F}_p) \longrightarrow TP_*(\mathbb{F}_p)$

are ring maps

can is explicit "inclusion".

We need to investigate φ.

Lem there exists $\alpha \in \mathbb{Z}_p^\times$ s.t.

$$\varphi(x) = \alpha t^{-1}, \quad \varphi(t) = \alpha^{-1} p t.$$

Rmk can show $\alpha = 1$.

Proof

$(THH(\mathbb{F}_p) \longrightarrow \mathbb{F}_p)^{tT}$

$t \mapsto \beta t$, $\quad \beta t \overset{\text{up to unit}}{\longmapsto} t \mapsto [\beta] t$

$$\begin{tikzcd}
TC^-(\mathbb{F}_p) \arrow[r, "\varphi"] \arrow[d] & TP(\mathbb{F}_p) \arrow[r] \arrow[d] & (\mathbb{F}_p)^{tT} \arrow[d] \simeq \mathbb{F}_p[t^\pm] \\
THH(\mathbb{F}_p) \arrow[r, "\varphi_p"] & THH(\mathbb{F}_p)^{tC_p} \arrow[r] & \mathbb{F}_p^{tC_p} \simeq \mathbb{F}_p[t^\pm] \otimes \Lambda(e)
\end{tikzcd}$$

$t \mapsto 0$ for deg reason

$t \mapsto [\beta] t$

$\mathbb{F}_2[s^\pm]$ $\quad p = 2$

Write $\varphi(x) = \alpha t^{-1}$, $\varphi(t) = \beta t$

diagram chasing gives $[\beta] = 0$.

so $\beta = p\beta'$.

Then $\varphi(xt) = p \Rightarrow \alpha\beta' = 1$. Done!

Now, consider LES for

$$TC \longrightarrow TC^{-} \xrightarrow{\varphi - \mathrm{can}} TP$$

$$\begin{tikzcd}
TC_2(\mathbb{F}_p) \arrow[r] & \mathbb{Z}_p x \arrow[r, "x \mapsto (1-\alpha^{-1}p)\alpha t^{-1}"] & \mathbb{Z}_p t^{-1} \arrow[dll] \\
TC_1(\mathbb{F}_p) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
TC_0(\mathbb{F}_p) \arrow[r] & \mathbb{Z}_p \arrow[r, "0"] & \mathbb{Z}_p \arrow[dll] \\
TC_{-1}(\mathbb{F}_p) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
TC_{-2}(\mathbb{F}_p) \arrow[r] & \mathbb{Z}_p t \arrow[r, "\alpha^{-1}p - 1"] & \mathbb{Z}_p t
\end{tikzcd}$$

Ex: $\mathbb{Z}_p \to \mathbb{Z}_p$, $x \mapsto x - p\alpha x$ is iso $\forall \alpha \in \mathbb{Z}_p$.

easy $(1-p\alpha)^{-1}$ converges.

Thm

$$TC_*(\mathbb{F}_p) = \begin{cases} \mathbb{Z}_p, & * = 0, -1 \\ 0, & \text{else} \end{cases}$$

Rmk - As a ring

$$TC_*(\mathbb{F}_p) = \mathbb{Z}_p[\varepsilon]/\varepsilon^2, \quad |\varepsilon| = -1$$

and indeed

$$TC(\mathbb{F}_p) = H\mathbb{Z}_p^{S^1}$$

as an $\mathbb{E}_\infty$-ring

- Quillen has shown that the p-completion of $K(\mathbb{F}_p)$ is

$$K(\mathbb{F}_p;\mathbb{Z}_p) = \pi_*((K\mathbb{F}_p)^\wedge_p)$$

$$= \begin{cases} \mathbb{Z}_p, & *=0 \\ 0, & \text{else} \end{cases}$$

$$\Rightarrow \quad K(\mathbb{F}_p;\mathbb{Z}_p) \simeq \tau_{\geq 0} TC(\mathbb{F}_p)$$

Below are its consequences.

Cor: There is an $\mathbb{E}_\infty$-map

$$H\mathbb{Z} \longrightarrow TC(\mathbb{F}_p)$$

- $TC(A)$ is an $H\mathbb{Z}$-module spectrum for each $\mathbb{F}_p$-alg A.

Proof: indeed we have

$$H\mathbb{Z}_p \longrightarrow TC(\mathbb{F}_p)$$

(connective cover)

- A is an $\mathbb{F}_p$-module in

$$\mathrm{Alg}(\mathrm{Ab}) \subset \mathrm{Alg}(\mathrm{Sp})$$

+ lax monoidal stuff

$$\mathbb{F}_p \otimes A \longrightarrow A$$

Prop We have an adjunction

$$(-)^{\mathrm{tr}} : \mathrm{Sp} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{CycSp} : \mathrm{TC}$$

The left adjoint is s.m.,
the right adjoint is lax s.m.

Cor . We have a map of cyclotomic $\mathbb{E}_\infty$-rings $H\mathbb{Z}^{\mathrm{triv}} \to \mathrm{THH}(\mathbb{F}_p)$

$\leftrightarrow$ $H\mathbb{Z} \to \mathrm{TC}(\mathbb{F}_p)$

. For A $\mathbb{F}_p$-alg,

$\mathrm{THH}(A)$ is naturally an $H\mathbb{Z}^{\mathrm{triv}}$-module

<u>Warning</u> The map $H\mathbb{Z} \xrightarrow{f} THH(\mathbb{F}_p)$ is different from the obvious map $g : H\mathbb{Z} \xrightarrow{p} H\mathbb{F}_p \longrightarrow THH(\mathbb{F}_p)$ as rings of spectra.

Indeed, for

$$r : THH(\mathbb{F}_p) \xrightarrow{\varphi_p} THH(\mathbb{F}_p)^{tC_p} \longrightarrow (\mathbb{F}_p)^{tC_p}$$

$$r \circ f \neq r \circ g .$$

(1) for f consider

$$
\begin{tikzcd}
H\mathbb{Z} \arrow[r, "\mathrm{triv}"] \arrow[d, "f"] \arrow[dd, bend right=60, "r\circ f"'] & H\mathbb{Z}^{tC_p} \arrow[d] \arrow[dl, phantom, ""] \arrow[ddl, bend left=40, "p^{tC_p}"] \\
THH(\mathbb{F}_p) \arrow[r, "\varphi_p"] & THH(\mathbb{F}_p)^{tC_p} \arrow[d] \\
 & \mathbb{F}_p^{tC_p}
\end{tikzcd}
$$

$$
rf \simeq \; H\mathbb{Z} \to H\mathbb{F}_p \xrightarrow{\mathrm{tr}} H\mathbb{F}_p^{tC_p}
$$

(2) for g

$$
\begin{tikzcd}
H\mathbb{F}_p \arrow[r, "\Delta_p"] \arrow[d] \arrow[ddr, bend right=40, "\mathbb{E}_\infty\text{-Frobenius}"'] & (H\mathbb{F}_p^{\otimes_{\mathbb{S}} p})^{tC_p} \arrow[d] \\
THH(\mathbb{F}_p) \arrow[r, "\varphi_p"] & THH(\mathbb{F}_p)^{tC_p} \arrow[d] \\
 & H\mathbb{F}_p^{tC_p}
\end{tikzcd}
$$

$$r_g \simeq H\mathbb{Z} \to H\mathbb{F}_p \xrightarrow[\mathbb{E}_\infty\text{-Frobenius}]{} H\mathbb{F}_p^{tC_p}$$

Thm The maps

$$H\mathbb{F}_p \xrightarrow{\text{triv}} (H\mathbb{F}_p)^{tC_p}$$

and the $\mathbb{E}_\infty$-Frobenius

$$H\mathbb{F}_p \to (H\mathbb{F}_p)^{tC_p}$$

differ by all the Steenrod operations

? see [NS]

Cor $f \neq g$.

Defn ∞-Cat of Cyclotomic chain complex

$\mathrm{Cyc}D\mathbb{Z}$ as the pullback

$$\begin{tikzcd}
\mathrm{Cyc}D\mathbb{Z} \arrow[r] \arrow[d] & \prod_p \left((D\mathbb{Z})^{BT}\right)^{\Delta^1} \arrow[d] \\
(D\mathbb{Z})^{BT} \arrow[r, "(\mathrm{id},\, (-)^{tC_p})_p"'] & \prod_p \left((D\mathbb{Z})^{BT} \times (D\mathbb{Z})^{BT}\right)
\end{tikzcd}$$

objects are like

$C \in D(\mathbb{Z})^{BT}$

$C \to C^{tC_p}$

Prop $\quad \mathrm{Cyc}(D\mathbb{Z}) \simeq \mathrm{Mod}_{H\mathbb{Z}^{\mathrm{triv}}}(\mathrm{CycSp})$

Some informal explanation:

$$H\mathbb{Z}^{\mathrm{triv}} \otimes (X, \varphi_p) \to (X, \varphi_p)$$

$$\begin{tikzcd} H\mathbb{Z} \otimes_{\mathbb{S}} X \arrow[r] \arrow[d] & X \arrow[dd] \\ H\mathbb{Z}^{tC_p} \otimes_{\mathbb{S}} X^{tC_p} \arrow[d] & \\ (H\mathbb{Z} \otimes_{\mathbb{S}} X)^{tC_p} \arrow[r] & X^{tC_p} \end{tikzcd}$$

$\rightsquigarrow$ $X \in D(\mathbb{Z})$

$X \in D(\mathbb{Z})^{BT}$

$X \to X^{tC_p}$

Cor For $\mathbb{F}_p$-alg A,

$\mathrm{THH}(A)$ is naturally a cyclotomic chain complex

but $\mathrm{HH}(A)$ isn't!