

$TC(F_P)$

Recall

$$TC(R) = E_{\mathbb{Z}} (TC^{\sim}(R) \xrightarrow[\text{can}]{\varphi} TP(R)^{\wedge})$$

for R discrete, bounded below

$$\text{can: } TC^{\sim}(R) \xrightarrow{\text{can}} TP(R) \rightarrow TP(R)^{\wedge}$$

$$\begin{aligned} \varphi: TC^{\sim}(R) &\xrightarrow{(\varphi_p)^{hT}} \prod_p (THH(R)^{tG_p})^{hT} \\ &= \prod_p (THH(R)^{tG})^{\wedge p} \\ &=: TP(R)^{\wedge} \end{aligned}$$

$$=: TP(R)^{\wedge}$$

(don't know why Thomas mentioned profinite

completion of TP here)

↑
looks like $(-)^{\wedge}$ has another meaning:

for G a discrete group, $\widehat{G} := \lim_{N \triangleleft G, [G:N] < \infty} G/N$

$$\text{e.g. } \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} \cong \prod_p \mathbb{Z}_p$$

Recall as proved before,

$$\cdot TC_*^{-1}(\mathbb{F}_p) = \mathbb{Z}_p[x, t] / xt - p,$$

$$|x| = 2, |t| = -2$$

$$\begin{aligned} \cdot TP_*(\mathbb{F}_p) &= TC_*^{-1}(\mathbb{F}_p)[t^{\pm 1}] \\ &= \mathbb{Z}_p[t^{\pm 1}] \end{aligned}$$

$$\Rightarrow TP(\mathbb{F}_p) = TP(\mathbb{F}_p)^\wedge.$$

$$\cdot \text{con. } \varphi : TC_*^{-1}(\mathbb{F}_p) \rightarrow TP_*(\mathbb{F}_p)$$

are ring maps

can is explicit "inclusion".

We need to investigate φ .

Lemma there exists $\alpha \in \mathbb{Z}_p^\times$ s.t.

$$\varphi(x) = \alpha t^{-1}, \quad \varphi(t) = \alpha^{-1} pt.$$

Remark can show $\alpha = 1$.

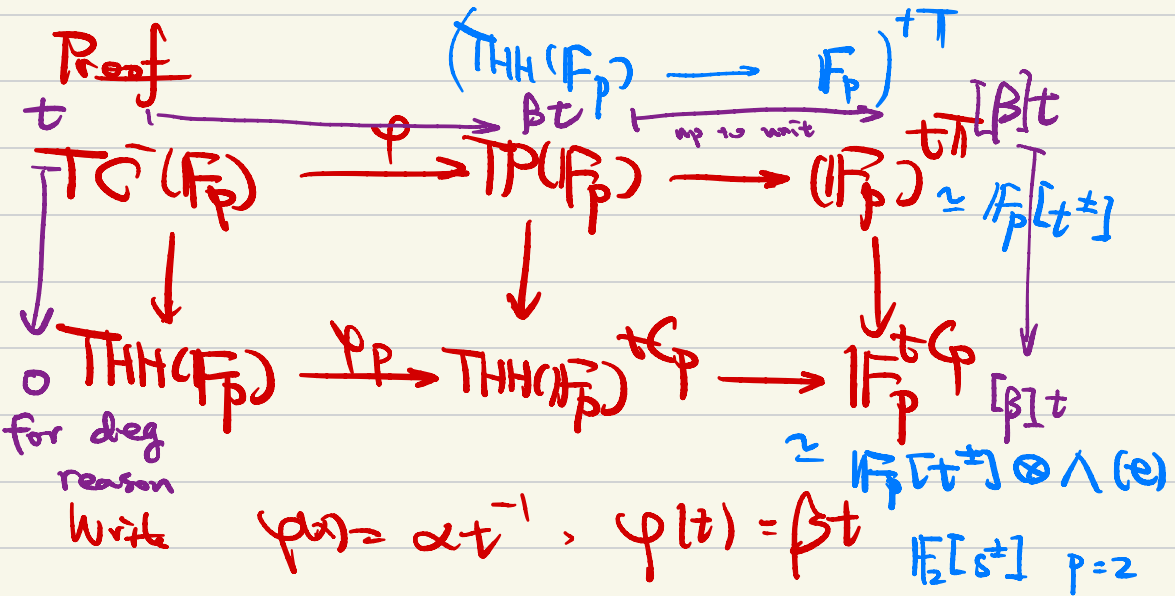


diagram chasing gives $[\beta] = 0$.

so $\beta = p\beta'$.

Then $\varphi(xt) = p \Rightarrow \alpha\beta' = 1$. Done!

Now, consider LES for

$$TC \rightarrow TC \xrightarrow{\varphi\text{-can}} TP$$

$$TC_2(\mathbb{F}_p) \rightarrow \mathbb{Z}_p x \xrightarrow{x \mapsto (1-x^p)x t^{-1}} \mathbb{Z}_p t^{-1}$$

$$\hookrightarrow TC_1(\mathbb{F}_p) \rightarrow 0 \rightarrow 0$$

$$\hookrightarrow TC_0(\mathbb{F}_p) \rightarrow \mathbb{Z}_p \xrightarrow{\circ} \mathbb{Z}_p$$

$$\hookrightarrow TC_{-1}(\mathbb{F}_p) \rightarrow 0 \rightarrow 0$$

$$\hookrightarrow TC_{-2}(\mathbb{F}_p) \rightarrow \mathbb{Z}_p t \xrightarrow{\alpha^{p-1}} \mathbb{Z}_p t$$

Ex: $\mathbb{Z}_p \rightarrow \mathbb{Z}_p$, $x \mapsto x - px$ is

iso $\forall \alpha \in \mathbb{Z}_p$.

easy $(1 - p\alpha)^{-1}$ converges.

Thm

$$TC_*(\mathbb{F}_p) = \begin{cases} \mathbb{Z}_p, & * = 0, -1 \\ 0, & \text{else} \end{cases}$$

Rule - As a ring

$$TC_*(\mathbb{F}_p) = \mathbb{Z}_p[\varepsilon]/\varepsilon^2, \quad |\varepsilon| = -1$$

and ideal

$$TC(\mathbb{F}_p) = H\mathbb{Z}_p \langle \delta^1 \rangle$$

as an $H\mathbb{F}_p$ -ring

- Quillen has show that the p -height of $K(\mathbb{F}_p)$ is

$$K(\mathbb{F}_p; \mathbb{Z}_p) = \prod_{*} (K(\mathbb{F}_p) \wedge_p^*) \\ = \begin{cases} \mathbb{Z}_p, & * = 0 \\ 0, & \text{else} \end{cases}$$

$$\Rightarrow K(\mathbb{F}_p; \mathbb{Z}_p) \simeq \mathbb{Z}_0 \subset C(\mathbb{F}_p)$$

Below are its consequences.

Con: There is an Eoo-map

$$HZ \longrightarrow TC(\mathbb{F}_p)$$

• $TC(A)$ is an $H\mathbb{Z}$ -module spectrum for each \mathbb{F}_p -alg A .

Proof: indeed we have

$$HZ_p \longrightarrow TC(\mathbb{F}_p)$$

(connective cover)

• A is an \mathbb{F}_p -module in

$$\text{Alg}(Ab) \subset \text{Alg}(Sp)$$

+ lex monoidal stuff

$$\mathbb{F}_p \otimes A \longrightarrow A$$

Prop We have an adjunction

$$(-)^{\text{tr}} : \mathcal{S}_p \rightleftarrows \text{Cyc} \mathcal{S}_p : \text{TC}$$

The left adjoint is s.m.,
the right adjoint is lax s.m.

Cor . We have a map of cyclotomic

$$\begin{aligned} \mathbb{E}_{\infty}\text{-rings} \quad \text{HZ}^{\text{triv}} &\longrightarrow \text{THH}(\mathbb{F}_p) \\ &\mapsto \text{HZ} \rightarrow \text{TC}(\mathbb{F}_p) \end{aligned}$$

• For A \mathbb{F}_p -alg .

$\text{THH}(A)$ is naturally an

HZ^{triv} -module

Warning The map $H\mathbb{Z} \xrightarrow{f} THH(\mathbb{F}_p)$

is different from the obvious

map $g: H\mathbb{Z} \xrightarrow{f} H\mathbb{F}_p \rightarrow THH(\mathbb{F}_p)$

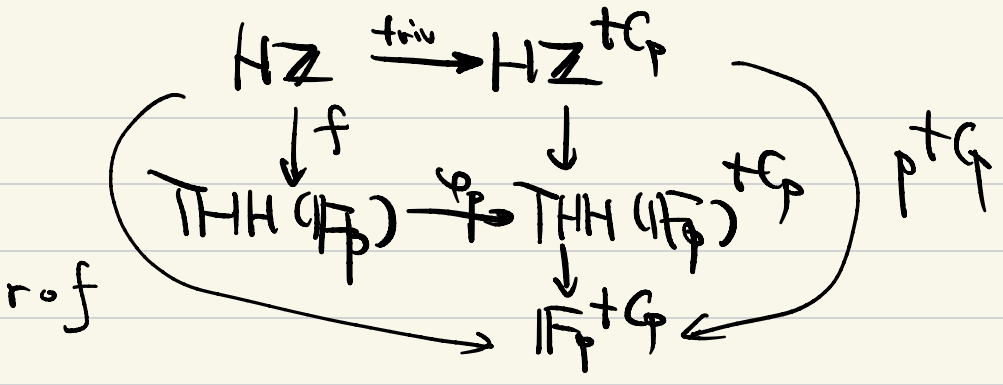
as rings of spectra.

Indeed, for

$$r: THH(\mathbb{F}_p) \xrightarrow{\varphi} THH(\mathbb{F}_p) \xrightarrow{+G} (\mathbb{F}_p)^{+G}$$

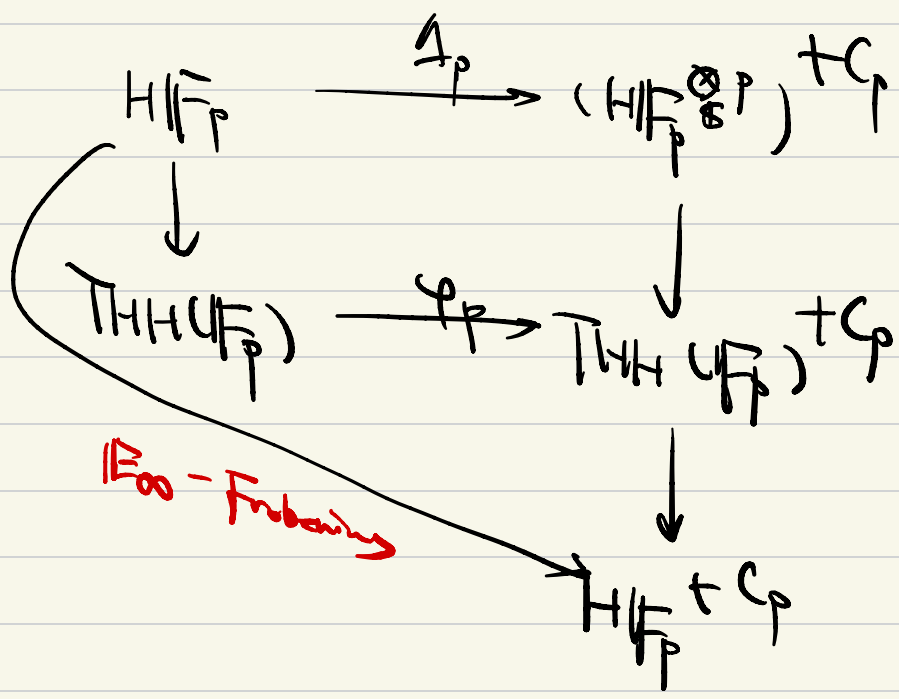
$$r \circ f \neq r \circ g.$$

(1) for f consider



$$r.f. \approx HZ \xrightarrow{triv} H\mathbb{F}_p \xrightarrow{triv} H\mathbb{F}_p^{+G}$$

(2) for g



$$r_g \cong \mathbb{H}\mathbb{Z} \xrightarrow{f} \mathbb{H}\mathbb{F}_p \xrightarrow[\text{Eo-Frobenius}]{g} \mathbb{H}\mathbb{F}_p^{tG}$$

Then The maps

$$\mathbb{H}\mathbb{F}_p \xrightarrow{\text{triv}} (\mathbb{H}\mathbb{F}_p)^{tG}$$

and the Eo - Frobenius

$$\mathbb{H}\mathbb{F}_p \xrightarrow{g} (\mathbb{H}\mathbb{F}_p)^{tG}$$

differ by all the Steenrod operations

? see [NS]

Con $f \neq g$.

Defn ∞ -Cat of Cyclotomic chain complex

CycDZ as the pullback

$$\begin{array}{ccc}
 \text{CycDZ} & \longrightarrow & \prod ((DZ)^{\text{BT}})^{\Delta^1} \\
 \downarrow & & \downarrow \\
 (DZ)^{\text{BT}} & \xrightarrow{(\text{id}, (-)^{+G_p})} & \prod ((DZ)^{\text{BT}} \times (DZ)^{\text{BT}})
 \end{array}$$

objects are like

$$C \in D(\mathbb{Z})^{\text{BT}}$$

$$C \rightarrow C^{+G_p}$$

$$\underline{\text{Prop}} \quad \text{Cyc}(\mathbb{D}\mathbb{Z}) \simeq \text{Mod}_{\mathbb{H}\mathbb{Z}^{\text{triv}}}(\text{CycSp})$$

Some informal explanation:

$$\mathbb{H}\mathbb{Z}^{\text{triv}} \otimes (X, \rho_p) \rightarrow (X, \rho_p)$$

$$\begin{array}{ccc} \boxed{\mathbb{H}\mathbb{Z} \otimes X \rightarrow X} & \rightsquigarrow & X \in \mathbb{D}(\mathbb{Z}) \\ \downarrow & & X \in \mathbb{D}(\mathbb{Z})^{\text{BT}} \\ \mathbb{H}\mathbb{Z} \otimes X^{\text{triv}} & & X \rightarrow X^{\text{triv}} \\ \downarrow & & \\ (\mathbb{H}\mathbb{Z} \otimes X)^{\text{triv}} & \rightarrow & X^{\text{triv}} \end{array}$$

Con For \mathbb{F}_p -alg A ,

$\overline{\text{THH}}(A)$ is naturally a
cyclotomic chain complex

but $\text{HH}(A)$ isn't!

