Properties of THH

<u>Recall</u> THH for $R \in \mathrm{Alg}(Sp)$.

- $k \in \mathrm{CAlg}(Sp)$, $R \in \mathrm{CAlg}(\mathrm{Mod}_k)$

$$\mathrm{THH}(R/k) := \mathrm{HH}(R/\mathrm{Mod}_k)$$

- $\mathrm{HH}(R) \simeq \mathrm{THH}(R/\mathbb{Z})$

$$\left(D(\mathbb{Z}) \simeq \mathrm{Mod}_{H\mathbb{Z}} \right)$$

<u>Prop</u> $\mathrm{THH} : \mathrm{Alg}(Sp) \longrightarrow Sp$ is s.m..

$\mathrm{HH}(-/\mathcal{C}) : \mathrm{Alg}(\mathcal{C}) \longrightarrow \mathcal{C}$ is s.m.

(for good $\mathcal{C}$)

<u>Proof</u> (holds intuitively, but has some coherence obstruction)

$$U : \mathrm{Alg}(Sp) \longrightarrow Sp$$

as an object in the s.m.

$$\mathrm{Fun}^{\otimes}(\mathrm{Alg}(Sp), Sp)$$

in fact it's an alg there

$$\mathrm{THH}(-) \simeq \mathrm{HH}\big(U/\underline{\mathrm{Fun}^{\otimes}(\mathrm{Alg}(Sp)}, \underline{Sp})\big)$$

geometric realizations exists and is computed underlyingly.

<u>Cor</u> $R \in \mathrm{Alg}_{E_n}(Sp)$ Sp can be replaced by $\mathcal{C}$

$\Rightarrow \mathrm{THH}(R) \in \mathrm{Alg}_{E_{n-1}}(Sp)$

Proof use $\mathrm{Alg}_{E_n} = \mathrm{Alg}_{E_{n-1}} \mathrm{Alg}_{E_1}$.

In particular, $n=\infty$. ...

$$A \underset{S}{\otimes} A = \operatorname*{colim}_{\{0,1\}} A = A^{\otimes 2}$$

<u>Recall</u> $A, B \in \mathrm{CAlg}(Sp)$,

$A \underset{S}{\otimes} B$ is the coproduct in $\mathrm{CAlg}(Sp)$.

<u>Prop.</u> For $R \in \mathrm{CAlg}(Sp)$

$$\mathrm{THH}(R) \cong \operatorname*{colim}_{S^1}(R) \in \mathrm{CAlg}(Sp)$$

$$=: R^{\otimes S^1}.$$

<u>Warning</u>: The colimit $\operatorname{colim}^{\mathrm{CAlg}(\mathrm{Sp})}_{S^1} R$ is different

from $\operatorname{colim}^{\mathrm{Sp}}_{S^1} R = \operatorname{colim}\{ \substack{\to\\\to\\\to} R\oplus R \rightrightarrows R\}$

$$\simeq R\otimes S^1$$

$$= R\otimes \Sigma^\infty_+ S^1$$

$$= R\oplus \Sigma R$$

<u>Base change formula for THH</u>

(1) given lax s.m. functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$

have a canonical

$$HH(FA/\mathcal{D}) \longrightarrow F(HH(A/\mathcal{C}))$$

for $A\in \mathrm{Alg}(\mathcal{C})$

(2) Suppose F strong s.m., preserves geo. rea.

then it's $\sim$.

Example: (1) $H : \mathcal{D}(\mathbb{Z}) \to \mathrm{Sp}$.

(2) Given $k \to k'$ of com. ring spectra.

$$- \otimes_k k' : \mathrm{Mod}_k \longrightarrow \mathrm{Mod}_{k'}$$

is sm., preserves colimits.

So $\mathrm{THH}(R/k) \otimes_k k' \simeq \mathrm{THH}(R \otimes_k k'/k')$

Now suppose k is a perfect, ordinary com. $\mathbb{F}_p$-alg.

e.g. $\mathbb{F}_{p^n}$, $\mathbb{F}_p[x^{1/p^\infty}] = \mathrm{colim}(\mathbb{F}_p[x] \xrightarrow{(-)^p} \mathbb{F}_p[x] \to \cdots)$

have $\quad Hk \to \mathrm{THH}(k) \qquad k \to \mathrm{THH}_*(k)$

$\quad \mathrm{THH}(\mathbb{F}_p) \to \mathrm{THH}(k) \qquad \mathbb{F}_p[x] \to \mathrm{THH}_*(k)$

$\rightsquigarrow Hk \otimes_{H\mathbb{F}_p} \mathrm{THH}(\mathbb{F}_p) \longrightarrow \mathrm{THH}(k)$, $\quad k[x] \to \mathrm{THH}_*(k)$.

Thm (Bökstedt periodicity for perfect rings)

k as above.

$k[x] \to THH_*(k)$ is iso

$\iff THH(k)$ is the free $\mathbb{E}_1$-alg on x over Hk.

Proof. k perfect $\Rightarrow \exists$ com. ring spectrum $\mathbb{S}_{W(k)}$ called "spherical Witt vectors"

s.t. $\mathbb{S}_{W(k)} \otimes_{\mathbb{S}} H\mathbb{F}_p \simeq k$

(lift of k along $\mathbb{S} \to H\mathbb{F}_p$)

k $\mathbb{F}_p$-alg $\rightsquigarrow$ $\mathbb{S}$-alg

$$\Rightarrow THH(k) \simeq THH(\mathbb{S}_{W(k)}) \otimes_{\mathbb{S}} THH(\mathbb{F}_p)$$

$$\simeq (THH(\mathbb{S}_{W(k)}) \otimes_{H\mathbb{F}_p} H\mathbb{F}_p) \otimes_{\mathbb{S}} THH(\mathbb{F}_p)$$

$$\simeq \underline{THH(k/H\mathbb{F}_p)} \otimes_{H\mathbb{F}_p} THH(\mathbb{F}_p)$$

$$HH(k/\mathbb{F}_p) \simeq k^{S^1}$$

For k $\mathbb{E}_\infty$-ring, $THH(k) = k^{\otimes S^1}$.

get an $\mathbb{E}_\infty$-map

$$THH(k) \longrightarrow k$$

$$\operatorname{colim}_{S^1}^{Alg(Sp)} k \longrightarrow \operatorname{colim}_{pt}^{Alg(Sp)} k$$

s.t. $k \longrightarrow THH(k) \longrightarrow k$

is id.

Rmk Such retract may not exist if

k is $\mathbb{E}_n$-ring spectrum $n < \infty$.

Thm k comm. ring spectrum

R $\mathbb{E}_1$ k-alg. (in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}_k)$).

Then $\mathrm{THH}(R/k) \simeq \mathrm{THH}(R) \otimes_{\mathrm{THH}(k)} k$

Sketch $\mathrm{colim}(\cdots \rightrightarrows R \otimes_k R \rightrightarrows R)$

$\simeq \mathrm{colim}(\cdots \rightrightarrows R \otimes_{\mathbb{S}} R \otimes_{k \otimes_{\mathbb{S}} k} R \rightrightarrows R)$

$\simeq \mathrm{colim}(\cdots \rightrightarrows R \otimes_{\mathbb{S}} R \rightrightarrows R) \otimes_{(\cdots \rightrightarrows k \otimes_{\mathbb{S}} k \rightrightarrows k)} (\cdots \rightrightarrows k \rightrightarrows k)$

$\simeq \mathrm{THH}(R) \otimes_{\mathrm{THH}(k)} k$