Period & Cyclic Homology

We know

$$HH(R/k) \in DGMod_A \longrightarrow DGMod_A[q.iso^{-1}] \simeq Mod_A(Dk)$$

$$(A = k[\delta]/\delta^2,\ |\delta| = 1,\ \partial = 0).$$

Defn.

(1) $$HC_*(R/k) := H_*(k \overset{\mathbb{L}}{\underset{A}{\otimes}} HH(R/k)) = \mathrm{Tor}_*^A(k, HH(R/k)).$$

(2) $$HC_*^-(R/k) := H_*(R\mathrm{Hom}_A(k, HH(R/k))) = \mathrm{Ext}_A^{-*}(k, HH(R/k)).$$

(Note. TC is more like HC^-)

It's a module over $\mathrm{Ext}^{-*}_A(k,k)$
$$\cong k[t],\ |t| = -2$$

$$\cdots \to k[t]/t^2 \xrightarrow{t} k[t]/t^2 \xrightarrow{t} k[t]/t^2 \to k$$
$$t \mapsto 0$$
$$1 \mapsto 1$$

looks a little like AdSS stuffs... maybe coincidence

(3) $$HP_*(R/k) = HC^-_*(R/k)[t^{-1}]$$

Note: I think HC_* should carry a gradation?

Prop. For any k-alg R, have

power series generated by t

$$HC^-(R/k) \cong (HH(R)[[t]],\ \partial + tB)$$
$$|t| = -2$$

where $\partial_t + tB$ is defined as

$$xt^i \longmapsto (\partial x)t^i + (Bx)t^{i+1}$$

$(x \in HH(R))$

Laurent power series

$$HP(R/k) \simeq (HH(R)((t)), \partial_t + B)$$

Proof Sketch of k: Consider A-alg resolution

$$\mathcal{C} = A\{x_0, x_1, \dots\}$$

$$|x_k| = 2k.$$

$$\partial x_k = b \cdot x_{k-1}.$$

(this is a coalg. w/

$$\Delta(x_k) = \sum_{i+j=k} x_i \otimes x_j.)$$

Rmk. This formula works more generally for any object $H \in DGMod_A$ to give

$$RHom_A(k, H).$$

Ob $\qquad A = k[b]/b^2.$

- A is a Hopf alg.

$$\varepsilon : A \to k, \quad \varepsilon(b) = 0.$$

$$\Delta : A \to A \otimes A, \quad \Delta(b) = 1 \otimes b + b \otimes 1$$

$\Rightarrow$ $\underline{RHom}_A(k, -)$ is a lax sim. functor as such, it is given by

$$C H [\![t]\!], \partial + tB)$$

→ If $\mathcal{C}$ is a alg. object in DGMod$_A$, i.e. $\mathcal{C}$ is DGA and B satisfies Leibniz rule, then $\underline{R\mathrm{Hom}}_A(k, \mathcal{C})$ is a DGA.

<u>Problem</u>. Even if R is com., B is NOT on the nose a derivation on $HH(R)$, but this is true <u>up to chain homotopy</u>. But this is enough to deduce

that $(HH(R)[\![t]\!], \partial + tB)$ has a product up to chain homotopy, i.e., that $HC^-_*(R)$ & $HP_*(R)$ are graded com. alg..

<u>**Defn**</u> R com. k-alg. The <u>de Rham cohomology</u> of R rel. to k is defined as

$$H^*_{dR}(R/k) := H_*(\Omega^*_{R/k}, d).$$

(this is indeed "boring")

<u>Rmk</u>. H^*_{dR} can be defined for

scheme X over k.

$\boxed{\text{Thm A}}$ Suppose $\mathbb{Q} \subset k$. R com. k-alg.

$L_{R/k}$ is flat dim 0.

Then have nat. iso.

$$HP_*(R/k) \cong H^*_{dR}(R/k)((t)), \ |t| = -2$$

$$HC^-_q(R/k) = Z^q_{dR}(R/k) \oplus \prod_{i \geq 1} H^{n+2i}_{dR}(R/k)$$

cycles

Rmk. For a scheme X over k w/

$L_{X/k}$ flat dim 0, have

$$HP_*(X/k) \cong H^*_{dR}(X/k)((t))$$

• False if char k is not zero.

only get a SS. ?

Consider:

$$\mu : HH(R/k)_n \longrightarrow \Omega^n_{R/k}$$

Lem. μ is a CDGA-map and A-linear

$$(HH(R/k), \partial, B) \to (\Omega^*_{R/k}, 0, d).$$

Cor. Suppose R has flat $L_{R/k}$ $\mathbb{Q} \subset k$.

then

$$\left(HH(R/k),\partial,B\right)\simeq\left(\Omega^*_{R/k},0,d\right)$$

(as DGA's & as A-modules)

<u>Proof</u>. We have

$$\Omega^*_{R/k}\xrightarrow{\text{can}}HH_*(R/k)\xrightarrow{H_*(\mu)}\Omega^*_{R/k}$$

the composite is identity,

so apply HKR. ✓

For <u>Thm A</u>,

$$HP_*(R/k)\cong H_*\left(HH(R/k)((t)),\partial+tB\right)$$

$$\cong H_*\left(\Omega^*_{R/k}((t)),td\right)$$

$$\simeq H^*_{dR}(R/k)((t))$$

$$HC^-_*(R/k) \simeq H_*(\Omega^*_{R/k}[[t]], td)$$

<u>Construction</u>. Consider Postnikov filtration

$$\tau_{\geq \cdot} HH(R/k)$$

on $HH(R/k)$ and leads to a filtration on $HP(R/k)$. Concretely,

$$\left(\tau_{\geq \cdot} HH(R/k)((t)), \partial + tB\right)$$

and leads to a multiplicative, conditional convergent SS

$$HH_*(R/k)((t)) \Rightarrow HP_*(R/k)$$

E_2-page:

$$H_*(HH_*(R/k), B)((t)) \Rightarrow HP_*(R/k)$$

If R has flat cotangent cplx, this is

$$E^3 = H^*_{dR}(R/k)((t)) \Rightarrow HP_*(R/k)$$

Defn R comm. ring, define the divided power series algebra $R\langle\langle x\rangle\rangle$ as the completion of $R\langle x\rangle$ at the filtration gen. by the divided power of x.

Prop: For $R = \mathbb{F}_p$,

$$HC_*^-(\mathbb{F}_p/\mathbb{Z}) \cong \mathbb{Z}_p[t]\langle\langle x\rangle\rangle/xt-p$$

$$|x| = |t| = 2.$$

$$HP_*(\mathbb{F}_p/\mathbb{Z}) \cong \mathbb{Z}_p[t^{\pm}]\langle\langle x\rangle\rangle/xt-p$$

$$\cong \left(\mathbb{Z}_p\langle\langle y\rangle\rangle/y-p\right)[t^{\pm}]$$

$$|y| = 0.$$

For top ver. divided power stuffs will go away.

Rmk $HP_0(\mathbb{F}_p/\mathbb{Z}) \cong HC_0^-(\mathbb{F}_p/\mathbb{Z})$

$\cong \mathbb{Z}_p\langle\langle y\rangle\rangle/y-p$

(obtained by adjoining divided power of p to $\mathbb{Z}_p$)

$\Rightarrow$ get torsion $y^{[p]} - p^p/p!$

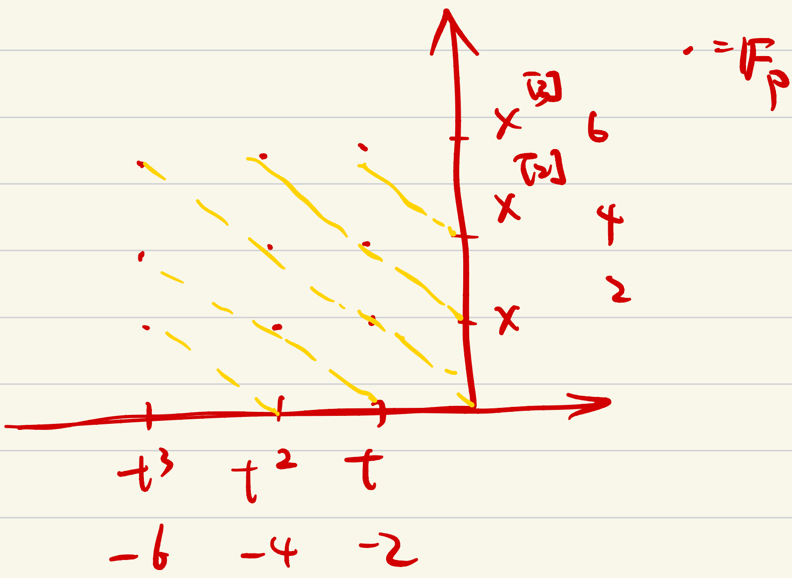
- In fact, have $\mathbb{Z}_p\{y\}/y-p \cong \mathbb{Z}_p(z)/z$

$z = y-p.$

- In fact, $HP_*(\mathbb{F}_p/\mathbb{Z})$ is 2-periodic derived de Rham cohomology of $\mathbb{F}_p$ rel $\mathbb{Z}$.

Proof. $HH_*(\mathbb{F}_p/\mathbb{Z}) \cong \mathbb{F}_p\langle x\rangle$.

SS: $\mathbb{F}_p\langle x\rangle[t] \Longrightarrow HC^-_*(\mathbb{F}_p/\mathbb{Z})$



collapses.

So $HC^-_*(\mathbb{F}_p/\mathbb{Z})$ has a fil. w/ graded given by $\mathbb{F}_p\langle x\rangle[t]$.

· The conn. cover of $HC^-(\mathbb{F}_p/\mathbb{Z})$ can be represented by a simplicial com ring. Thus it admits

divided powers on positive degree

htpy groups.

In particular, any choice of

$x, t \in HC^-_*(\mathbb{F}_p/\mathbb{Z})$ gives a map

$$\mathbb{Z}\langle x\rangle[t] \longrightarrow HC^-_*(\mathbb{F}_p/\mathbb{Z}).$$

Want to find x, t s.t. $xt = p$ in HC^-_*,

then this map induced an iso on ass.

graded of the map

$$\mathbb{Z}\langle x\rangle[t]/xt-p \longrightarrow HC^-_*(\mathbb{F}_p/\mathbb{Z})$$

In the computation of $HH(\mathbb{F}_p/\mathbb{Z})$

$$\mathbb{F}_p \simeq (\Lambda_{\mathbb{Z}}(\varepsilon),\ \partial\varepsilon = p)$$

In $HH(-)$, $x = B\varepsilon$, $\partial x = 0$, $Bx = 0$.

x represents $x \in HH_*(\mathbb{F}_p/\mathbb{Z})$
$= \mathbb{F}_p\langle x \rangle.$

$$HC^-(\mathbb{F}_p) = (HH(\mathbb{F}_p/\mathbb{Z})[[t]], \partial + tB)$$

$(\partial + tB)\varepsilon = p + tx \Rightarrow p = tx$
in $HC_*^-(\cdots)$.

Rmk One can also deduce this computation using

$$(HH(\mathbb{F}_p/\mathbb{Z}), \partial, B) \simeq \left(\mathbb{Z}[\varepsilon]/\varepsilon^2 \langle x \rangle, \begin{array}{ll} \partial\varepsilon = p & B\varepsilon = x \\ \partial x = 0 & Bx = 0 \end{array}\right)$$

- One can construct LES:

$$\cdots \to HC_{*-1}(R) \to HC_*^-(R/k) \to HP_*(R/k) \to$$