TC

Defn Recall $THH(R)$ carries an S^1-action.

$$TC^-(R) = THH(R)^{hS^1}$$

Defn (stable ∞-cat)...

Lem If $\mathcal{C} \in \mathrm{Cat}_\infty^{ex}$, then there is a unique lift

$$\begin{tikzcd} & & Sp \arrow[d, "\Omega^\infty"] \\ \mathcal{C}^{op} \times \mathcal{C} \arrow[rr, "\mathrm{Map}"] \arrow[urr, dashed, "\mathrm{map}"] & & \mathcal{S} \end{tikzcd}$$

such that map is exact in both variables.

Proof sketch

In $\mathcal{C}$, $Y \simeq \Omega\Sigma Y \simeq \Omega^2\Sigma^2 Y \simeq \cdots$

$$\mathrm{Map}(X, Y) \simeq \Omega\,\mathrm{Map}(X, \Sigma Y)$$
$$\simeq \Omega^2\,\mathrm{Map}(X, \Sigma^2 Y)$$
$$\simeq \cdots$$

defines a spectrum.

Lem. I ∞-cat. Then

- $\mathrm{Fun}(I, \mathcal{S}p)$ is stable
- $\lim_I F = \mathrm{map}_{\mathrm{Fun}(I, \mathcal{S}p)}(\mathrm{const}\,\mathbb{S}, F)$

can be verified by using adjn of $\mathrm{map}(-,-)$.

Prop $HC^-(R) \simeq HH(R)^{hS^1}$

Proof identify $\mathcal{D}(\mathbb{Z})$ w/ $\mathrm{Mod}_{H\mathbb{Z}}$ (as s.m. ∞-cat)

$$HH(R)^{hS^1} \simeq \mathrm{map}_{\mathrm{Fun}(BS^1, \mathrm{Sp})}(\mathbb{S}, HH(R))$$

limits in $\mathrm{Mod}_{H\mathbb{Z}}$ are calculated in Sp.

$$\simeq \mathrm{map}_{\mathrm{Fun}(BS^1, \mathrm{Mod}_{H\mathbb{Z}})}(H\mathbb{Z}, HH(R))$$

$$\simeq \mathrm{map}_{\mathrm{Fun}(BS^1, \mathcal{D}(\mathbb{Z}))}(\mathbb{Z}, HH(R))$$

$$\simeq \mathrm{map}_{\mathrm{Mod}_A(\mathcal{D}(\mathbb{Z}))}(\mathbb{Z}, HH(R))$$

$$\simeq \mathrm{RHom}_A(\mathbb{Z}, HH(R))$$

$$=: HC^-(R)$$

Note: $\mathrm{Fun}(BS^1, \mathcal{D}(\mathbb{Z})) \xrightarrow{\sim} \mathrm{Mod}_A(\mathcal{D}(\mathbb{Z}))$ is not s.m., so, though $HC^-(R)$ will have a good DGA struct., it's not given by

$R\mathrm{Hom}_A(\mathbb{Z}, HH(R))$.

Lem $(-)^{hS^1} : \mathrm{Fun}(BS^1, Sp) \to Sp$ is lax s.m..

and the S^1-action on $THH(R)$ (or $HH(R)$) is compatible w/ the product struct. (if R is $\mathbb{E}_\infty$)

Cor There's an $\mathbb{E}_\infty$-struct. on $TC^-(R)$ if R is $\mathbb{E}_\infty$ (compatibly on $HC^-(R)$)

Lem Let $BS^1 \to Sp$ be an S^1-action on HA, so A is merely a discrete abelian group; then

$$\pi_*(HA)^{hS^1} = A[t] \quad \text{w/ } |t| = -2$$
$$\shortparallel$$
$$A \otimes \mathbb{Z}[t]$$

Proof The full subcat of Sp on all EM spectra (discrete) is

1-Cat:

$$\mathrm{Map}(HA, HB) \simeq \mathrm{Hom}(A, B)$$

it's $Sp^{\heartsuit} \simeq Ab$.

a functor $BS^1 \to Sp$, $* \to HA$

must be constant.

So $$HA^{hS^1} \simeq \mathrm{map}_{Sp}(\Sigma^{\infty}_{+} BS^1, HA)$$

$$\pi_*(HA^{hS^1}) \cong H^*(BS^1, A)$$

$BS^1 \simeq K(\mathbb{Z}, 2) \simeq \mathbb{C}P^{\infty}$.

<u>Thm</u> (HFPSS)

homotopy fixed point SS

There is a multiplicative & conditionally convergent SS $\quad |t| = -2$

$$\pi_*(X)[t] \Rightarrow \pi_*(X^{hS^1})$$

<u>Proof sketch</u> Consider the fil..

Whitehead tower:

$$\begin{tikzcd} \tau_{\geq 2} X \arrow[d] \\ \tau_{\geq 1} X \arrow[r] \arrow[d] & \Sigma H\pi_1 X \\ \tau_{\geq 0} X \arrow[r] \arrow[d] & \Sigma H\pi_0 X \\ \vdots \end{tikzcd} \rightsquigarrow \begin{tikzcd} (\tau_{\geq 2} X)^{hS^1} \arrow[d] \\ (\tau_{\geq 1} X)^{hS^1} \arrow[r] \arrow[d] & (H\pi_1 X)^{hS^1} \\ (\tau_{\geq 0} X)^{hS^1} \arrow[r] \arrow[d] & (H\pi_0 X)^{hS^1} \\ \vdots \end{tikzcd}$$

(of spectra w/ S^1-action)

not Whitehead tower of X^{hS^1}!

We need the right tower to be complete, exhaustive, etc. ..

An S^1-action on X gives a map

$$\Sigma^\infty_+ S^1 \otimes X \longrightarrow X$$

S^1-action on spectra coincides w/ module struct. given by the ring $\Sigma^\infty_+ S^1$.

$$\Sigma^\infty_+ S^1 \xrightarrow{\text{can}} \Sigma^\infty_+ pt \simeq \mathbb{S}$$

splits, so have a splitting

$$\Sigma^\infty_+ S^1 \simeq \mathbb{S} \oplus \Sigma^\infty S^1$$

$$\simeq \mathbb{S} \oplus \Sigma \mathbb{S}$$

$\rightsquigarrow$ $\Sigma X = \Sigma \mathbb{S} \otimes X \longrightarrow X$, i.e.

a map $b : \pi_*(X) \longrightarrow \pi_{*+1}(X)$

generalization of Connes operator

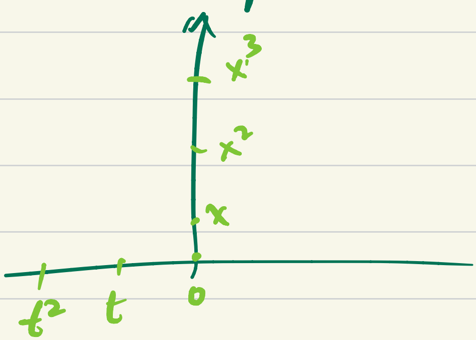
<u>Lem</u> In the HFPSS, d_2 is determined by

$$d_2 \alpha = b\alpha \cdot t \ , \ \alpha \in \pi_* X$$

$$d_2 t = \eta t^2$$

try: χ be the free module over $\Sigma^\infty_+ S^1$.

Example $TC^-(\mathbb{F}_p)$



1) $TC^-_{2k+1}(\mathbb{F}_p) = 0$

$TC^-_{2k}(\mathbb{F}_p)$ are complete filtered abelian groups, w/ associated graded given by a sequence of $\mathbb{F}_p$'s.

2) $TC_0^-(\mathbb{F}_p) \longrightarrow \mathbb{F}_p$

is surject w/ kernel generated

as ideal by $\tilde{t}\tilde{x}$

$\tilde{t} \in TC_{-2}^-$, $\tilde{x} \in TC_2^-$ lifts of t, x

so there is a relation $p = a\tilde{t}\tilde{x}$

3) the map $THH(\mathbb{F}_p) \longrightarrow HH(\mathbb{F}_p)$

is iso on $\tau_{\leq 2}$, so

$$(\tau_{\leq 2} THH(\mathbb{F}_p))^{hS^1} \xrightarrow{\sim} (\tau_{\leq 2} HH(\mathbb{F}_p))^{hS^1}.$$

saw in SS for HC^-: $\tilde{t}\tilde{x} = p$.

so a is a unit

Modify choice of $\tilde{x}, \tilde{t}$, WLOG $a = 1$.

so $\tilde{t}\tilde{x} = p$.

$$\rightsquigarrow \quad \mathbb{Z}_p[\tilde{t}, \tilde{x}]/(p - \tilde{t}\tilde{x}) \longrightarrow TC_*^-(\mathbb{F}_p)$$

<u>Thm</u> $\quad TC_*^-(\mathbb{F}_p) \cong \mathbb{Z}_p[\tilde{x}, \tilde{t}]/(\tilde{x}\tilde{t} - p)$

<u>Cor</u> $\quad TC_{2k}^-(\mathbb{F}_p) \cong \mathbb{Z}_p$

generated by $\begin{cases} \tilde{x}^{-k}, & \text{if } k \geq 0, \\ \tilde{t}^{-k}, & \text{if } k \leq 0, \end{cases}$