HKR & Cotangent Complex

$\mathcal{C}$ 1-cat, gen by cpt proj. objects

Hom(K, −) detects iso

Hom(K, −) preserves reflexive coequalizers $\cdot \rightrightarrows \cdot$

$$\mathrm{Ani}(\mathcal{C}) = \mathrm{Fun}^{\Pi}((\mathcal{C}^{cp})^{op}, \mathcal{S})$$

- $\mathcal{C}^{cp} \subseteq \mathrm{Ani}(\mathcal{C})$ full subcat (full subset)

$\mathrm{Ind}(\mathcal{C}^{cp}) \subseteq \mathrm{Ani}(\mathcal{C})$

- general objects can be represented as geometric realizations of

Simplicial diagrams with entries in $\mathrm{Ind}(\mathcal{C}^{op})$

$$- \mathrm{Map}_{\mathrm{Anim}(\mathcal{C})}\Big(\overset{X \text{ cpt proj.}}{X}, \operatorname*{colim}_{j \in \Delta^{op}} Y_j\Big)$$

$$= \operatorname*{colim}_{j \in \Delta^{op}} \mathrm{Map}_{\mathrm{Anim}(\mathcal{C})}(X, Y_j)$$

means that $\mathrm{Ani}(\mathcal{C})$ can be viewed as formal geo realizations of simplicial objects in $\mathrm{Ind}(\mathcal{C}^{op})$.

<u>Example</u>: $D(A)_{\geq 0} \simeq \mathrm{Ani}(A)$

have enough cpt proj.

non additive $\rightsquigarrow F : A \to B \ (\to \mathrm{Ani}(B))$

extends to a filt. colimit + geom. rea. preserving functor

$$\mathrm{Ani}(A) \longrightarrow \mathrm{Ani}(B)$$

non-abelian ver. of derived functor

<u>Example</u> R com. .

$$\Lambda^n_R : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$$

(non-additive in general)

$$\rightsquigarrow$$

$$\mathbb{L}\Lambda^n_R : \mathcal{D}(R)_{\geq 0} \longrightarrow \mathcal{D}(R)_{\geq 0}$$

computed by representing objects by simplicial diagrams of proj. modules, and then levelwise applying Λ^n_R.

<u>Exercise</u> : 1. Construct a $\underbrace{\text{levelwise projective}}$ simplicial abelian

group quasi-iso to $\mathbb{Z}/n[0]$.
(under Dold-Kan)

2. Compute $\mathbb{L}\Lambda^2_{\mathbb{Z}}(\mathbb{Z}/n)$.

For 1, first resolve $\mathbb{Z}/n[0]$ as $\mathbb{Z}[\varepsilon]/\varepsilon^2$, $d\varepsilon = n$, $|\varepsilon| = 1$.

Then consider its Dold-Kan correspondence $DK(\mathbb{Z}[\varepsilon]/\varepsilon^2)$, namely,

$$DK(\mathbb{Z}[\varepsilon]/\varepsilon^2)_q = \bigoplus_{[q] \twoheadrightarrow [k]} (\mathbb{Z}[\varepsilon]/\varepsilon^2)_k$$

$$= \mathbb{Z} \oplus \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_q.$$

$d_i : DK_q \to DK_{q-1}$, $1 \le i \le q-1$.

is defined by $1 \mapsto 1$,

$$\mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_q \to \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_{q-1}$$

repeats $\mathbb{Z}\varepsilon_i$ and "nondecreasing"

$d_0 : \varepsilon_1 \mapsto n$, $\varepsilon_j \mapsto \varepsilon_{j-1}$, $(2 \le j \le q)$

$d_n : \varepsilon_q \mapsto 0?$ $\varepsilon_j \mapsto \varepsilon_j$, $(1 \le j \le q-1)$

$s_i : DK_q \longrightarrow DK_{q+1}, \quad 0 \leq i \leq q$

satisfies $\quad 1 \longmapsto 1$

" increasing " - skip $\mathbb{Z}\Sigma_{q-1}$

2.

$$\begin{array}{c}
\vdots \\
\mathbb{Z}\alpha \oplus \mathbb{Z}\Sigma_1 \oplus \mathbb{Z}\Sigma_2 \\
\downarrow\uparrow\downarrow\uparrow\downarrow \\
\mathbb{Z}\alpha \oplus \mathbb{Z}\Sigma_1 \\
d_0 \downarrow\uparrow\downarrow d_1 \\
\mathbb{Z}\alpha
\end{array}
\xrightarrow{\Lambda^2_{\mathbb{Z}}} \xrightarrow{N}
\begin{array}{cc}
\downarrow & \\
0 & \\
\downarrow & \\
\mathbb{Z}\Sigma_1\Sigma_2 & 2 \\
\downarrow n & \\
\mathbb{Z}\alpha\Sigma_1 & 1 \\
\downarrow & \\
0 & 0
\end{array}$$

(can be shown $N\Lambda^2_{\mathbb{Z}}$ (this guy) vanishes when deg > 2)

So $\quad \mathbb{L}\Lambda^2_{\mathbb{Z}}(\mathbb{Z}/n[0]) \simeq \mathbb{Z}/n[1]$.

$$\mathrm{Anim}(\mathrm{cRing}) = \mathrm{Fun}^{\pi}((\mathrm{cRing}^{\omega})^{op}, \mathcal{S}]$$

$$= \mathrm{Fun}^{\pi}((\mathrm{Poly}^{fg})^{op}, \mathcal{S})$$

$$\mathrm{Ind}(\mathrm{cRing}^{\omega}) = \mathrm{Poly}$$

represented by simplicial diagrams of polynomial rings

$$\mathrm{Map}_{\mathrm{Anim}(\mathrm{cRing})}(\mathbb{Z}[X], \operatorname{colim}_{\Delta^{op}} Y_i) = \operatorname{colim}_{\Delta^{op}} Y_i$$

$$\mathrm{Fun}^{\mathrm{geom+filt}}(\mathrm{Anim}(\mathrm{cRing}), \mathcal{D})$$

$$\simeq \mathrm{Fun}(\mathrm{Poly}^{fg}, \mathcal{D})$$

$= \mathrm{Fun}^{\mathrm{fil}}(\mathrm{Poly}, \mathcal{D})$

(accessible functors $\mathrm{Poly} \to \mathcal{D}$)

Example The functor

$$\mathrm{HH}(-/\mathbb{Z}) : \mathrm{Poly} \longrightarrow \mathcal{D}(\mathbb{Z})_{\geq 0}.$$

commutes w/ fil. colimits,

so extends to

$$\mathbb{L}\mathrm{HH} : \mathrm{Anim}(\mathrm{cRing}) \longrightarrow \mathcal{D}(\mathbb{Z})_{\geq 0}.$$

on $\mathrm{cRing} \to \mathrm{Anim}(\mathrm{cRing})$

this agrees w/ $\mathrm{HH}(R/\mathbb{Z})$

? agrees for all animated rings

by considering the associated DGA

Construction

For $\mathcal{C} \in \mathcal{D}(\mathbb{Z})$, let $\tau_{\geq n}\mathcal{C} \to \mathcal{C}$

w/ · Iso on H_i, $\forall i \geq n$

· $H_i(\tau_{\geq n}\mathcal{C}) = 0$, $i < n$.

it's a right adjoint to $D(\mathbb{Z})_{\geq n} \hookrightarrow D(\mathbb{Z})$?

$$\mathrm{Cofib}(\tau_{\geq n+1}\mathcal{C} \to \tau_{\geq n}\mathcal{C}) \simeq (H_n\mathcal{C})[n]$$

For R polynomial ring:

$$\begin{array}{c} \vdots \\ \downarrow \\ \tau_{\geq n+1} HH(R/\mathbb{Z}) \\ \downarrow \\ \tau_{\geq n} HH(R/\mathbb{Z}) \to \Omega^n_{R/\mathbb{Z}}[n] \\ \downarrow \\ \vdots \\ \downarrow \end{array}$$

$$HH(R/\mathbb{Z}) \simeq \tau_{\geq 0}(HH(R/\mathbb{Z}))$$

Def

$$F^n_{HKR} HH(-/\mathbb{Z}) : \mathrm{Anim}(\mathrm{CRing}) \longrightarrow D(\mathbb{Z})_{\geq 0}$$

as geom. realization preserving extension.

if $\tau_{\geq n} HH(-/\mathbb{Z})$ from

$$\mathrm{Poly} \longrightarrow D(\mathbb{Z})_{\geq 0}$$

$\tau_{\geq n}$ commutes w/ fil. colimits.

Here

$$F^{n+1}_{HKR} HH(R/\mathbb{Z}) \longrightarrow F^n_{HKR} HH(R/\mathbb{Z}) \longrightarrow L\Omega^n_{R/\mathbb{Z}}[n]$$

Lem

$$F^n_{HKR} HH(R/\mathbb{Z}) \in D(\mathbb{Z})_{\geq n}$$

Lem. Suppose

$F : \mathrm{cRing} \to \mathrm{Ab}$ com. w/

reflective coequalizers.

then

$$H_0(\mathbb{L}F(R)) = F(R)$$

Proof resolve R: $R_0 \leftleftarrows R_1 \, \leftleftarrows\!\!\!\leftarrow R_2 \cdots$

Observe: $R = \mathrm{Coeq}(R_0 \leftleftarrows R_1)$

$LF(R) = \mathrm{Tot}(F(R_0) \leftleftarrows F(R_1) \leftleftarrows\!\!\!\leftarrow \cdots)$

$$H_0 LF(R) = \mathrm{Coeq}(F(R_0) \leftleftarrows F(R_1))$$
$$= F(R)$$

Ex. $R \mapsto R^{\times n}$: $\mathrm{cRing} \to \mathrm{Set}$

$R \mapsto \mathbb{Z}[R^{\times n}]$: $\mathrm{cRing} \to \mathrm{Ab}$

$\Omega^i_{-/\mathbb{Z}}$: $\mathrm{cRing} \to \mathrm{Ab}$

commutes w/ reflexive coequalizers.

HKR ver. 2

If $L\Omega^n_{R/\mathbb{Z}}$ is concentrated in deg 0, $(\forall n)$, then HKR holds for R.

Proof sketch Consider HKR fil..

Ob $L\Omega^n_{R/\mathbb{Z}}$ agrees w/ the value on R of

$$\mathrm{Anim}(\mathrm{CRing})_{/R} \longrightarrow D(R)_{\geq 0} \longrightarrow D(\mathbb{Z})_{\geq 0}$$

$$A \in \mathrm{Poly}_{/R} \longmapsto R \otimes_A \Omega^n_{A/\mathbb{Z}}.$$

$$R \otimes_A \Omega^n_{A/\mathbb{Z}} \simeq \Lambda^n_R \big(\underbrace{R \otimes_A \Omega^1_{A/\mathbb{Z}}}\big)$$

takes c.p. objects in $k\,\mathrm{Alg}_{/R}$ to c.p. objects in Mod_R

have $$L\Omega^n_{R/\mathbb{Z}} = L\Lambda^n_R(L\Omega^1_{R/\mathbb{Z}})$$

Prop if $L\Omega_{R/\mathbb{Z}}$ concentrates in deg 0, and $\Omega^1_{R/\mathbb{Z}}$ is a flat R-mod, then $L\Omega^n_{R/\mathbb{Z}}$ also concentrates in deg 0.

Lazard: every flat R-mod is a fil. colimit of f.g. proj. R-mod.

Thm HKR. Same assumption as above. Then
$$HH_n(R/\mathbb{Z}) \simeq \Omega^n_{R/\mathbb{Z}}$$