HH

k field, R k-alg.

$$HH_*(R/k) = H_* \, HH(R/k).$$

$$HH(R/k) = \left(\cdots \xrightarrow{\partial} R \otimes_k R \xrightarrow{\partial} R \right).$$

given by cyc bar construction

$$\cdots R^{\otimes_k 3} \; \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} \; R^{\otimes_k 2} \; \substack{\rightarrow\\ \leftarrow\\ \rightarrow} \; R$$

Example:

$$HH_*(k/k) = \begin{cases} k, & * = 0 \\ 0, & \text{otherwise.} \end{cases}$$

R com..

Kähler differential

$$HH_0(R/k) = R$$

$$HH_1(R/k) \simeq R^{\otimes 2}/\sim \;\cong\; \Omega^1_{R/k}$$

$$x \otimes y \longmapsto x\,dy.$$

<u>Lem</u>. R Com.

Then

$HH_*(R)$ has struct. of a strictly graded Com. ring.

<u>Proof</u>. Applying Eilenberg-Zilberg to the cyc bar construction.

Defn $\quad \Omega^*_{R/k} = \Lambda^*_R \Omega^1_{R/k}$.

Thm (HKR)

If R/k has cotangent complex concentrated in deg 0,

$$\Omega^*_{R/k} \xrightarrow{\text{can}} HH_*(R/k) \quad \text{is iso..}$$

determined by deg $= 0, 1$

on deg 0: $R \longrightarrow R$

deg 1: $\Omega^1_{R/k} \to HH_1(R/k)$

$x\,dy \mapsto [x \otimes y]$.

k com. ring

R dga over k, k-flat

$$HH(R/k) = \text{total complex}\left(\cdots \to R \otimes_k R \to R \right)$$

For R not k-flat, consider a k-flat resolution R^b.

$$HH(R/k) := HH(R^b/k)$$

Prop For a dga R

$$HH(R/k) \simeq R \otimes^{\mathbb{L}}_{R \otimes^{\mathbb{L}}_k R^{op}} R$$

not classically referred to as Hochschild homology, where $R \otimes^{\mathbb{L}} R^{op}$ should be $R \otimes_k R^{op}$.

Proof replace R by flat resolution

$$\underbrace{\mathrm{Tot}\left(\cdots \to R \otimes_k R \otimes_k R \to R \otimes_k R \right)}_{\text{bar complex}} \simeq R$$

Apply $- \otimes_{R \otimes_k R^{op}} R$ becomes cyc Bar

the total complex obtained computes

$R \otimes^{L}_{R \otimes_k R^{op}} R$ and $HH(R/k)$

Lem If R is a alg coming from a simplicial com. k-alg, then $HH(R/k)$ also comes from $\cdots$.

in particular, $HH_*(R/k)$ from strictly graded com. k-alg.

<u>Lem</u> A, B dga, have

$$HH(A \overset{\mathbb{L}}{\otimes} B/k) \simeq HH(A/k) \underset{k}{\overset{\mathbb{L}}{\otimes}} HH(B/k)$$

Prop $HH_*(\mathbb{F}_p/\mathbb{Z}) \simeq \mathbb{F}_p\langle x \rangle$.

Appendix for simplicial com. rings.

Def

$$\mathrm{sCRing} = \mathrm{Fun}(\Delta^{op}, \mathrm{CRing})$$

Thm. (Eilenberg - Zilberg)

$$\mathrm{sAb} \to \mathrm{Ch}$$

is lax s.m. ..

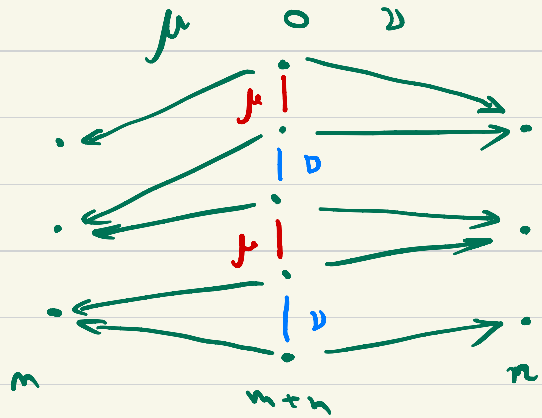
Cor have an induced

$$\mathrm{sCRing} \longrightarrow \mathrm{cdga}$$

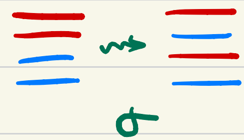
(m, n) - shuffle:

$$(\mu, \nu) : [m+n] \longrightarrow [m] \times [n].$$

- μ, ν both surj.
- μ, ν 'jump at disjoint positions



$m = n = 2$



formally, $\sigma \in \Sigma_{m+n}$ w/
$\sigma(1) < \cdots < \sigma(m)$
$\sigma(m+1) < \cdots < \sigma(m+n)$

- $\mathrm{sgn}(\mu, \nu) = \mathrm{sgn}(\sigma)$

Prop. $R \in \mathrm{ScRing}$, then

$$R_m \otimes R_n \xrightarrow{\ \cdot\ } R_{m+n}$$

$$x \cdot y = \sum_{\substack{(\mu,\nu) \\ \in (m,n)\text{-shuffle}}} \mathrm{sign}(\mu, \nu)\, S_\mu(x) \cdot S_\nu(y).$$

gives R the struct. of a cdga.

$x^2 = 0$ for total deg x.

Prop $R \in \mathrm{ScRing}$.

Then one can construct "divide powers"

$$\gamma_k : R_n \longrightarrow R_{nk} \quad \text{for } n, k \geq 1$$

s.t.

- $\gamma_1(x) = x$ $\qquad x^k = k!\,\gamma_k(x)$
- $\gamma_k(x)\gamma_\ell(x) = \binom{k+\ell}{k}\gamma_{k+\ell}(x)$
- $\gamma_k(x+y) = \sum_{i+j=k} \gamma_i(x)\gamma_j(y)$.
- $\gamma_k(xy) = x^k\gamma(y)$
- $\gamma_k(\gamma_\ell(x)) = \frac{(k\ell)!}{k!(\ell!)^k}\gamma_{k\ell}(x)$.

__Proof sketch__ Show that x^k is "divisable" by $k!$.

Indeed, for x odd, $\gamma_k(x) = 0$ for $k \geq 2$.

for x even,

$$\gamma_k(x) = \sum_{\substack{(\mu_1, \dots, \mu_k) \in \\ \text{shuffle}/\Sigma_k}} \operatorname{sgn}(\mu_1, \dots, \mu_k)\, s_{\mu_1}(x) \cdots s_{\mu_k}(x).$$

Prop $R \in \text{SCRing}$.

- $d\gamma_k(x) = \gamma_{k-1}(x)\, dx$ if x even
- γ_k sends boundaries to boundaries
- γ_k give well-defined divided power structure on $H_*(R)$.