HH of Schemes

X scheme over k
ring

Want to define $HH(X/k)$

Approaches

(1) Extended from affine schemes

(2) Generalize $HH(-/k)$ to dg-cats over k and define

$$HH(X/k) = HH(\mathrm{Perf}(X)/k).$$

"non com. approach".

Def. For a scheme X,

$Sch_{/k}^{op} \to D(k)$.

$$HH(X/k) = \lim_{\substack{U \subset X \\ U \text{ open affine}}} HH(\mathcal{O}(U)/k) \in D(k)$$

right extension.

$HH(X/k)$ may not be connective

$HH(\mathcal{O}(-)/k)$:

$$\{\text{affine opens in } X\}^{op} \longrightarrow D(k).$$

(agree w/ ordinary HH

as $HH(\operatorname{Spec} R/k) = \lim_{U \subset \operatorname{Spec} R} HH(\mathcal{O}(U)/k)$

$\overset{\operatorname{Spec} R \text{ initial}}{\cong} HH(\mathcal{O}(\operatorname{Spec} R)/k)$

$\cong HH(R/k)$)

Could go for stacks as well.

<u>Example</u> : $\mathbb{P}^1_k$

covered by two $\mathbb{A}^1_k$ (glue along $x \mapsto \frac{1}{y}$)

$(\mathbb{A}^1_k)^+ = \operatorname{Spec} k[x]$, $(\mathbb{A}^1_k)^- = \operatorname{Spec} k[y]$.

$(\mathbb{A}^1_k)^+ \cap (\mathbb{A}^1_k)^- = \mathbb{G}_m = \operatorname{Spec}(k[x^{\pm 1}])$

Descent / Mayer-Vietoris sequence?

<u>Thm</u> $U, V \subset X$ subschemes, $U \cup V = X$, then have

$$\begin{tikzcd} HH(X/k) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & HH(U/k) \arrow[d] \\ HH(V/k) \arrow[r] & HH(U \cap V/k) \end{tikzcd}$$

in $D(k)$.

Rmk Will see $HH(-/k)$ satisfies even flat ($=$ fpqc) descent.

Cor. In the above thm, have LES

$\cdots \longrightarrow \cdots$ (M-V seq).

Lem $$HH(R[x^{-1}]/k) \cong HH(R/k) \otimes_R R[x^{-1}].$$

(R com. $x \in R$)

Indeed, we have flat base change formula:

$$HH(T, T \otimes_R M \otimes_R T) \simeq T \otimes_R HH(R, M)$$

for $R \to T$, T flat as R-mod.

$$HH(S^{-1}R, S^{-1}R) \simeq HH(R, S^{-1}R) \simeq S^{-1} HH(R, R)$$

$$T \otimes^{L}_{T^e} (T^e \otimes_{R^e} M) \simeq T \otimes^{L}_{T^e} (T^e \otimes^{L}_{R^e} M) \quad (T^e \text{ flat } R^e\text{-mod})$$

$$\simeq T \otimes^{L}_{R^e} M$$

$$\simeq T \otimes^{L}_{R} (R \otimes^{L}_{R^e} M)$$

$$\simeq T \otimes_{R} (R \otimes^{L}_{R^e} M) \quad (T \text{ flat } R\text{-mod})$$

Prop One can use M-V seq for

$(\mathbb{P}^1_k ; (\mathbb{A}^1_k)^+, (\mathbb{A}^1_k)^-)$ to calculate

$HH_*(\mathbb{P}^1_k/k)$. $\cdots\cdots$

Proof of this pullback square:

can assume WLOG that X is affine.

invoking some quasi-compactness can assume U, V affine as well.

then $\boxed{\mathrm{I}}$ $\iff$

$$\begin{tikzcd}
HH(R/k) \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp"] & HH(R[x^{-1}]/k) \arrow[d] \\
HH(R[y^{-1}]/k) \arrow[r] & HH(R[x^{-1}, y^{-1}]/k)
\end{tikzcd}$$

$HH(R/k) \otimes_R R[x^{-1}]$

$HH(R/k) \otimes_R R[y^{-1}]$ $\quad$ $HH(R/k) \otimes_R R[x^{-1}, y^{-1}]$

$\Leftarrow$

$$\begin{tikzcd}
R \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp"] & R[x^{-1}] \arrow[d] \\
R[y^{-1}] \arrow[r] & R[x^{-1}, y^{-1}]
\end{tikzcd}$$

in $D(k)$

Now is cotangent complex.

Recall we have HKR filtration

$$F^*_{HKR} HH(R/k) := \mathbb{L}(\tau_{\geq *} : kAlg \to D(k)_{\geq 0})$$

w/ associated graded

$$F^{n+1}_{HKR} / F^n_{HKR} = \mathbb{L}\Omega^n_{R/k}[n].$$

This fil. is complete:

$$\lim F^n_{HKR} HH(R/k) = 0.$$

Now for X scheme over k.

$$F^*_{HKR} HH(X/k) = \lim_{U \subset X} F^*_{HKR} HH(\mathcal{O}(U)/k).$$

Prop $F^*_{HKR} HH(X/k)$ is complete fil.. of $HH(X/k)$, w/ assoc. graded

$$\lim_{U \subset X} L\Omega^n_{\mathcal{O}(U)/k}[n] = R\Gamma(X; L\Omega^n_{\mathcal{O}/k}[n])$$

(the derived Hodge cohomology of X)

$R\Gamma(X; \Omega^n_{\mathcal{O}/k}[n])$ is the

sheaf cohomology of $\Omega^n_{\mathcal{O}/k}[n]$, the

(non-derived) Hodge cohomology

In particular we get

Hodge-Hochschild spectral sequence

$$R\Gamma^p(X, L\Omega^k_{\mathcal{O}/k}[q]) \Rightarrow HH_{p+q}(X/k)$$

$\boxed{\text{Cor}}$ Suppose $\mathbb{Q} \subset k$.

Then $$HH(X/k) \simeq \prod_n R\Gamma(X, L\Omega^n_{\mathcal{O}/k}[n])$$

(in fact compatible w/ B)

Proof True if $X = \mathbb{A}^n_k$.

also true for affine X.

general case follows.? by non-abelian derived functor?