HH in ∞-Cat

<u>Recall</u> $\mathrm{Assoc}^{\otimes}_{\mathrm{act}}$

$$\left(= \mathrm{Env}(\mathrm{Assoc}^{\otimes}) \right).$$

<u>Defn</u> An assoc. alg in a s.m. ∞-Cat. is given by a s.m. functor

$$N(\mathrm{Assoc}^{\otimes}_{\mathrm{act}}) \longrightarrow \mathcal{C}.$$

$\mathrm{Assoc}^{\otimes}_{\mathrm{act}}$ is in some sense determined by $\langle 1 \rangle$ — which is an assoc algebra (or a monoid)

The underlying object is the value at $\langle 1 \rangle$.

Exercise $\{\text{s.m. functor } \mathrm{Assoc}^{\otimes}_{\mathrm{act}} \to \mathrm{Ab}\} \simeq \mathrm{Ring}$

Recall: $\mathrm{Cut}: \Delta^{op} \to \mathrm{Set}$

Cut^{cyc}: $\mathrm{Cut}^{cyc}(S) = \mathrm{Cut}(S) / (S, \emptyset) \sim (\emptyset, S)$.

Exercise $\mathrm{Cut} = \Delta^1$,

$\mathrm{Cut}^{cyc} = \Delta^1 / \partial\Delta^1$.

$$\mathrm{Cut}^{cyc}: \Delta^{op} \longrightarrow \mathrm{Assoc}^{\otimes}_{\mathrm{act}}$$

for $f: S \to T$.

$(S_0, S_1) \in \mathrm{Cut}^{cyc}(S)$.

if $(S_0, S_1) \neq (\emptyset, S) \sim (S, \emptyset)$.

$(f^{-1})^*(S_0, S_1) = \{\text{cuts between } f(S_0), f(S_1)\}$

if $(S_0, S_1) = (\emptyset, S) \sim (S, \emptyset)$.

$(f^{-1})^*(S_0, S_1) = \{\text{cuts outside } f(S)\}$.

Def $A \in \mathrm{Alg}(\mathcal{C})$.

$$HH(A/\mathcal{C}) := \operatorname{colim}\left(\Delta^{op} \xrightarrow{\mathrm{Cut}^{cyc}} \mathrm{Assoc}_{\mathrm{act}}^{\otimes} \xrightarrow{A} \mathcal{C}\right).$$

Lem A an ordinary ring (or dga),

have $A \in \mathrm{Alg}(\mathcal{D}(\mathbb{Z}))$.

Then $HH(A/\mathcal{D}(\mathbb{Z})) \simeq HH(A/\mathbb{Z})$

Defn $A \in \mathrm{Alg}(\mathrm{Sp})$.

$$THH(A) := HH(A/\mathrm{Sp})$$

for $R \in \mathrm{Alg}(\mathrm{Ab})$,

H is lax s.m.

$$HR \in \mathrm{Alg}(\mathrm{Sp})$$

$$THH(R) := THH(HR)$$

Example $THH(\mathbb{S}) = \mathbb{S}$

Def. $\mathrm{LMod}_{act}^{\otimes}$

- objects: finite sets colored by $\{a, m\}$
- maps: maps $S \to T$

w/ total ordering on each preimage.

s.t. • pre-image of a-colored elements are completely a-colored

• pre-img of m-colored elements contains contains precisely one m-colored element as the max.

$Env(LMod)$

A left module in $\mathcal{C}$ is a s.m. functor $LMod^{\otimes}_{act} \to \mathcal{C}$

(so {left module in $\mathcal{C}$} = $Alg_{LMod}(\mathcal{C})$)

$$\begin{tikzcd} LMod^{\otimes}_{act} \arrow[r] & \mathcal{C} \\ Assoc^{\otimes}_{act} \arrow[u, hook] & \end{tikzcd}$$

Define a cat $LRMod^{\otimes}_{act}$

- objects finite sets colored by $\{r, a, l\}$.

- maps $f: S \to T$ w/ ordering on

each pre-image s.t.

- pre-image of a-colored elements are a-colored
- pre-image of r-colored elements has precisely one r-colored element as the min, rest being a-colored.
- ---- (l-colored)

A s.m. $\mathrm{LRMod}^{\otimes}_{act} \to \mathcal{C}$ is a pair of $\mathrm{LMod}^{\otimes}_{act} \to \mathcal{C}$

$\mathrm{RMod}^{\otimes}_{act} \to \mathcal{C}$

which agree on $\mathrm{Assoc}^{\otimes}_{act}$.

Defn $N \otimes_A M =$

$$\mathrm{Colim}\left(\Delta^{op} \xrightarrow{\mathrm{Cut}(-)} \mathrm{LRMod}^{\otimes}_{act} \xrightarrow{(N,A,M)} \mathcal{C}\right)$$

not Cut^{cyc}!

r : (ϕ, S)

l : (S, ϕ)

a : others

- One can define $\mathrm{BMod}^{\otimes}_{\mathrm{act}}$ in an analogous way
  - $\{a, m\}$ colored finite set
  - pre-image of a-colored are a-colored

    of m-colored consists exactly

  of one m-colored, no other requirement

$\mathrm{Cut}^{\mathrm{cyc}}$ factors through $\mathrm{BMod}^{\otimes}_{\mathrm{act}}$,

one can define $HH(A/\mathcal{C}; M)$.

$$\left| \cdots \substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow} A \otimes A \otimes M \substack{\rightarrow\\ \rightarrow\\ \rightarrow} A \otimes M \rightrightarrows M \right|.$$

Recall $\mathrm{Comm}^{\otimes}_{\mathrm{act}} = \mathrm{Fin} \simeq \mathrm{Env}(\mathrm{Fin}_*)$.

Recall $\mathrm{CAlg}(\mathcal{C})$.

Lem $A \in \mathrm{CAlg}(\mathcal{C})$,

then $HH(A/\mathcal{C})$ has a com. alg struct..

$$H : \mathcal{D}(\mathbb{Z}) \longrightarrow Sp$$

Prop

- $\pi_* H(\mathcal{C}) = H_*(\mathcal{C})$.
- H is lax s.m.
- colimit-preserving

We get a canonical map

$$THH(HR) \longrightarrow H(HH(R)).$$

base change for $H : \mathcal{D}(\mathbb{Z}) \to Sp$

$$\begin{tikzcd} & & Sp \\ \Delta^{op} \arrow[r] & \mathrm{Assoc}^{\otimes}_{act} \arrow[ur, "HR"] \arrow[r, "R"'] \arrow[r, phantom, shift left=4, "\Downarrow"] & \mathcal{D}(\mathbb{Z}) \arrow[u, "H"'] \end{tikzcd}$$

As a Cor: $THH_*(HR) \xrightarrow{can} HH_*(R)$.

$H : \mathcal{D}(\mathbb{Z}) \longrightarrow Sp$

canonically factored thru $\mathcal{D}(\mathbb{Z}) \xrightarrow[\text{s.m.}]{\sim} Mod_{H\mathbb{Z}}$.

so $H(HH(R)) = HH(HR/\mathrm{Mod}_{H\mathbb{Z}})$
$\simeq THH(HR/H\mathbb{Z})$

Example. For $R \in \mathbb{Q}\text{-Alg}$,

have $THH_*(R) \xrightarrow{\sim} HH_*(R)$.

as $HR \otimes_{\mathbb{S}} HR \simeq HR \otimes_{H\mathbb{Z}} HR$.

Prop. R ordinary ring.

$$THH_i(R) \to HH_i(R)$$

is iso for $i \leq 2$, surj for $i = 3$.
i.e. 3-connected.

Sketch Consider $\mathrm{fib}(THH(R) \to THH(HR/H\mathbb{Z}))$

Thm (Bökstedt)

$$THH_*(\mathbb{F}_p) = \mathbb{F}_p[x], \quad |x| = 2.$$

Note that $\mathbb{F}_p[x] \to \mathbb{F}_p\langle x\rangle$ is zero in deg $\geq 2p$