Frobenius Lifts

and

Group Rings

Def A p-cyclotomic spectrum w/ Frobenius lift is a p-cyclotomic ? spectrum X, w/ a $\mathbb{T}$-eqvt factorization

$$X \xrightarrow{\varphi_p} X^{hC_p} \xrightarrow{\mathrm{can}} X^{tC_p}$$

or rather C_{p^∞}-eqvt

of $\varphi_p : X \to X^{tC_p}$.

which extends automatically to $\mathbb{T}$-case if X is p-complete, bounded below [NS] P121.

Note: A p-cyclotomic spectrum is a spectrum w/ C_{p^∞}-action X w/ C_{p^∞}-eqvt map

$X \to X^{tC_p}$ (under $C_{p^\infty}/C_p \overset{\mathrm{can}}{\simeq} C_{p^\infty}$)

where $C_{p^\infty} \subset \mathbb{T}$ is the subgroup of elements of p-power torsion. $C_{p^\infty} \simeq \mathbb{Q}_p/\mathbb{Z}_p \simeq \mathbb{Q}/\mathbb{Z}_{(p)}$

$\simeq \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \simeq p^{-\infty}\mathbb{Z}/\mathbb{Z}$

and we have a corresponding TC for p-cyclotomic spectra.

<u>Example</u> X^{triv} admits a Frobenius lift

$$\varphi_p : X \xrightarrow{\psi_p} X^{hC_p} \longrightarrow X^{tC_p}$$

$\mathrm{map}(\Sigma^\infty_+ pt, X)$ $\mathrm{map}(\Sigma^\infty_+ BC_p, X)$

+ bounded below

If X has a Frobenius lift and is p-complete

$\varphi : X^{h\mathbb{T}} \longrightarrow X^{t\mathbb{T}}$ factors as

$$\begin{tikzcd} X^{h\mathbb{T}} \arrow[r, "\psi_p^{h\mathbb{T}}"] \arrow[dr, "\varphi"'] & (X^{hC_p})^{h\mathbb{T}} \arrow[r, "\mathrm{triv}^{h\mathbb{T}}"] \arrow[d, phantom, "\simeq"] & (X^{tC_p})^{h\mathbb{T}} \arrow[d, phantom, "\simeq"] \\ & X^{h\mathbb{T}} \arrow[r, "\mathrm{can}"] & X^{t\mathbb{T}} \end{tikzcd}$$

$\simeq$ Tate orbit lemma

more exactly, see [NS] Lem II.4.2

<u>Recall</u> : For X a spectrum w/ C_{p^2}-action bounded below, then $(X_{hC_p})^{t(C_{p^2}/C_p)} \simeq 0$

A (much) further <u>Cor</u> :

Indeed $(X^{tC_p})^{hT} \simeq (X^{tT})^{\wedge}_p$

X p-complete $\Rightarrow$ X^{tT} p-complete?

So $\varphi - \mathrm{can} : X^{hT} \longrightarrow X^{tT}$ factors

as $$X^{hT} \xrightarrow{\varphi - id} X^{hT} \xrightarrow{\mathrm{can}} X^{tT}$$

<u>Thm</u> X p-complete, bounded below, p-cyclotomic spectrum w/ Frobenius lift, then

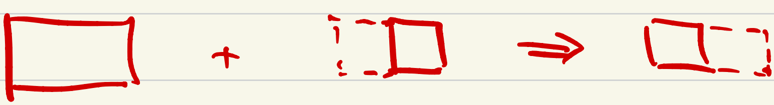
$$\begin{tikzcd}
TC(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Sigma X_{hT} \arrow[d, "tr"] \arrow[r] & X^{hT} \arrow[dl] \\
X \arrow[r, "\varphi_p - id"] \arrow[dr] & X & \\
& X^{hC_p} \arrow[ur] &
\end{tikzcd}$$

Proof Sketch

$$
\begin{tikzcd}
TC(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp"] & 0 \arrow[d] \\
X^{h\mathbb{T}} \arrow[r, "\varphi - can"] & X^{t\mathbb{T}}
\end{tikzcd}
$$

$$
\begin{tikzcd}
TC(X) \arrow[r] \arrow[d] & \Sigma X_{h\mathbb{T}} \arrow[r] \arrow[d, "Nm"] \arrow[dr, phantom, "\perp"] & 0 \arrow[d] \\
X^{h\mathbb{T}} \arrow[r, "\psi - id"'] \arrow[d] \arrow[dr, phantom, "\perp"] & X^{h\mathbb{T}} \arrow[r, "can"] \arrow[d] & X^{t\mathbb{T}} \\
X \arrow[r, "\psi_p - id"] & X &
\end{tikzcd}
$$

[NS], Section 4



Cor

$$TC(X^{triv}) = X \oplus \Big(X \otimes \underbrace{\mathrm{fib}\big(\Sigma \mathbb{S}_{h\mathbb{T}} \xrightarrow{tr} \mathbb{S}\big)}_{\Sigma \mathbb{CP}^{\infty}_{-1}} \Big)$$

as $\psi_p - id = 0$

$\mathbb{CP}^{\infty}_{-1}$ thought of "having a cell at each $2k$ deg, $k \geq -1$"

Let G be an $\mathbb{E}_1$-group in $\mathcal{S}$.

Write $\mathbb{S}[G] := \Sigma^{\infty}_{+} G$

Rmk $K(\mathbb{S}[G])$ is interesting, it's called the A-theory of $Y \simeq BG$, which is important in geometric topology.

Lem $$\mathrm{THH}(\mathbb{S}[G]) \simeq \Sigma^\infty_+ LBG$$

free loop space

i.e.

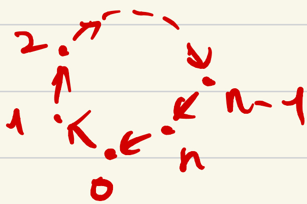
$\mathrm{Map}(S^1, BG)$.

Proof $$\mathrm{THH}(\mathbb{S}[G]) = \Sigma^\infty_+ |B^{cyc}(G)|$$

Compare $B^{cyc}(G)$ w/ the cyclic object

$$\mathrm{Fun}(S_\bullet, BG).$$

where S_n is the 1-cat freely generated by



$\mathrm{Fun}(S_n, BG)$ is not $G^{\times n}$, but $\left(G^{\times n}\right)_{h(G^{\times n})}$,

So $\left|\mathrm{Fun}(S_\bullet, BG)\right|$ is obtained from

$\left|B^{cyc}(G)\right|$ by taking orbits under

$$\left| \cdots \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} G\times G\times G \substack{\rightarrow\\\rightarrow\\\rightarrow} G\times G \rightrightarrows G \right| \simeq pt$$

Now $\mathrm{Fun}(S_n, BG)$ ← groupoid

$= \mathrm{Map}(|S_n|, BG)$

$\simeq \mathrm{Map}(S^1, BG)$

$\simeq LBG$

Proof (1) The $\mathbb{T}$-action on

$$THH(\mathbb{S}[G]) \simeq \Sigma^\infty_+ LBG$$

comes from the $\mathbb{T}$-action on LBG

given by rotating loops

(2) The Frobenius

$$\Sigma^\infty_+ LBG \longrightarrow \Sigma^\infty_+ LBG^{tC_p}$$

admits a lift

$$\Sigma^\infty_+ LBG \longrightarrow \Sigma^\infty_+ LBG^{hC_p}$$

coming from the C_p-invariant map

$$LBG \longrightarrow LBG$$

that is precomposition w/ the deg p

map $S^1 \to S^1$.

Key insight this prop:

the Tate diagonal $\mathbb{S}[G] \longrightarrow (\mathbb{S}[G]^{\otimes C_p})^{tC_p}$

lift through the diagonal $G \to G^{\times p}$.

Prop idea comes from [Bökstedt–Carlsson–Cohen–Goodwillie–Hsiang–Madsen]

For G an $\mathbb{E}_1$-group w/ $\pi_0 G$ a p-group

$$\begin{tikzcd} \Sigma^\infty_+ LBG \arrow[r, "\psi_p - \mathrm{id}"] \arrow[d, "ev_1"] & \Sigma^\infty_+ LBG \arrow[d, "ev_1"] \\ \Sigma^\infty_+ BG \arrow[r, "0"] & \Sigma^\infty_+ BG \end{tikzcd}$$

is a pull-back after p-completion.

Proof cofibres of horizontal maps are $(\Sigma^\infty_+ LBG)_{hN}$, $(\Sigma^\infty_+ BG)_{hN}$

$$\Sigma^\infty_+ (\psi_p^{-1} LBG)_{h\mathbb{Z}} \simeq (\Sigma^\infty_+ LBG)_{hN}, \qquad \Sigma^\infty_+ (BG)_{h\mathbb{Z}} \simeq (\Sigma^\infty_+ BG)_{hN}$$

it suffices to check $\psi_p^{-1} LBG \longrightarrow BG$ is an $\mathbb{F}_p$-homology equiv

$$\begin{tikzcd}
G \arrow[r, "p"] \arrow[d] & G \arrow[r, "p"] \arrow[d] & G \arrow[r] \arrow[d] & \cdots & p^{-1}G \arrow[d] \\
LBG \arrow[r, "\psi_p"] \arrow[d] & LBG \arrow[r, "\psi_p"] \arrow[d] & LBG \arrow[r] \arrow[d] & \cdots & \psi_p^{-1} LBG \arrow[d] \\
BG \arrow[r, "id"] & BG \arrow[r, "id"] & BG \arrow[r] & \cdots & BG
\end{tikzcd}$$

Suffices to check $H_\cdot(p^{-1}G, \mathbb{F}_p) = 0$

$p^{-1}G \simeq p^{-1}G_0$

Now, $H_*(p^{-1}G_0 ; \mathbb{F}_p)$ is a connected Hopf algebra, and multiplying by p induces

$$H_*(G_0) \xrightarrow{\Delta} H_*(G_0)^{\otimes p} \xrightarrow{\mu} H_*(G_0)$$

Ex. $A \xrightarrow{\Delta} A^{\otimes p} \xrightarrow{\mu} A$ is degreewise nilpotent for A a connected Hopf algebra over $\mathbb{F}_p$.

Cor For G an $\mathbb{E}_1$-group w/ $\pi_0 G$ a p-group,

$$\begin{tikzcd} TC(\mathbb{S}[G]; \mathbb{Z}_p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \left(\Sigma(\Sigma^\infty_+ LBG)_{hT}\right)^\wedge_p \arrow[d] \\ (\Sigma^\infty_+ BG)^\wedge_p \arrow[r, "0"] & (\Sigma^\infty_+ BG)^\wedge_p \end{tikzcd}$$

i.e.

$$TC(\mathbb{S}[G]; \mathbb{Z}_p) \simeq \Big(\Sigma^\infty_+ BG \oplus \mathrm{fib}\big(\Sigma(\Sigma^\infty_+ LBG)_{hT} \xrightarrow{tr} \Sigma^\infty_+ LBG \xrightarrow{-ev_1} \Sigma^\infty_+ BG\big)\Big)^\wedge_p$$