$\mathbb{E}_n$ – Algebras

$\mathcal{C}^{\otimes}$ s.m. ∞-cat.

$\mathrm{Alg}(\mathcal{C}) := \mathrm{Fun}^{\otimes}(\mathrm{Assoc}^{\otimes}_{\mathrm{act}}, \mathcal{C})$

$\mathrm{CAlg}(\mathcal{C}) := \mathrm{Fun}^{\otimes}(\mathrm{Comm}^{\otimes}_{\mathrm{act}}, \mathcal{C})$.

<u>Warning</u>: for 1-cat $\mathcal{C}$,

$$\mathrm{CAlg}(\mathcal{C}) \overset{\text{fact-uhfue}}{\subseteq} \mathrm{Alg}(\mathcal{C}).$$

This is false in ∞-cat case.

though there is a can.

$$\mathrm{CAlg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C}).$$

Def of $(\mathbb{E}_n^{\otimes})_{act}$ $(0 \le n < \infty)$

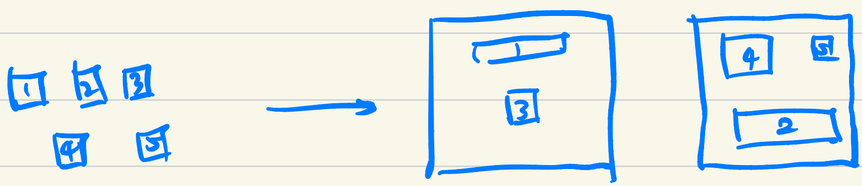
Rmk: Indeed $\mathbb{E}_n^{\otimes}$ can be defined in a similar "pointed" manner.

$$(\mathbb{E}_n^{\otimes})_{act} = N(-)$$

obj. $\underbrace{D^n \amalg \cdots \amalg D^n}_{0 \le k \text{ fold}}$, $D^n = (0,1)^n$

maps $D^n \amalg \cdots \amalg D^n \longrightarrow D^n \amalg \cdots \amalg D^n$

are given by rectilinear embeddings



This cat is top enriched & s.m. (by $\amalg$)

Def. An $\mathbb{E}_n$-Algebra in $\mathcal{C}$ is a s.m. functor

$$(\mathbb{E}_n^{\otimes})_{act} \longrightarrow \mathcal{C}$$

$$\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) := \mathrm{Fun}^{\otimes}((\mathbb{E}_n)^{\otimes}_{\mathrm{act}}, \mathcal{C}).$$

<u>Rmk</u> used a "nontrivial fact", Top is "disjunctive".

<u>Rmk</u> Mapping spaces in $(\mathbb{E}_n^{\otimes})_{\mathrm{act}}$ are given by

$$\begin{array}{c} \mathrm{Map}_{(\mathbb{E}_n^{\otimes})_{\mathrm{act}}}(D^n_{\underline{k}} = \amalg D^n, D^n) \\ \Big\downarrow \text{ ev at centres} \\ \mathrm{Conf}_k(D^n) \end{array}$$

it's a h.e.

<u>Prop</u> There is an equiv.

$$\mathrm{Assoc}^{\otimes}_{\mathrm{act}} \longrightarrow \mathbb{E}_1^{\otimes}$$

of s.m. ∞-cats

$$S \longmapsto \coprod_S D^1.$$

Ex 1. Works out $\mathrm{Alg}_{\mathbb{E}_0}(\mathcal{C})$.

2. Works out $\mathrm{Alg}_{\mathbb{E}_2}(1\text{-Cat})$.

$$\mathbb{E}^{\otimes}_{0,\mathrm{act}} = \mathrm{Fin}^{\mathrm{inj}}$$

1. An informal examination suggests

$$\mathrm{Alg}_{\mathbb{E}_0}(\mathcal{C}) = \mathcal{C}^{1/}.$$

2. Braided monoidal cat.

We have functors

$$(\mathbb{E}^{\otimes}_0)_{\mathrm{act}} \xrightarrow{\times D^1} (\mathbb{E}^{\otimes}_1)_{\mathrm{act}} \xrightarrow{\times D^1} (\mathbb{E}^{\otimes}_2)_{\mathrm{act}} \longrightarrow \cdots$$

Thus

$$\cdots \longrightarrow \mathrm{Alg}_{\mathbb{E}_2}(\mathcal{C}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{C})$$

Defn $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}) := \lim \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$

Thm $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathcal{C})$.

Let $(\mathcal{C}, \otimes)$ be closed s.m. or weaker

assume that $- \otimes c : \mathcal{C} \to \mathcal{C}$ commutes

w/ filtered colimits and geometric

realizations & $\mathcal{C}$ has all limits

and colimits

Thm: (1) $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$ admits all

limits & colimits,

and

$$\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \xrightarrow[ev_{D^n}]{fgt} \mathcal{C}$$

preserves filtered colimits, geometric realizations & and all limits.

Moreover it detects equivs.

(2) For $n=\infty$, the coproduct is given by $A \otimes B$, $A, B \in \mathrm{CAlg}(\mathcal{C})$

(3) The ∞-cat $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$ a s.m. struct s.t.

$$\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \longrightarrow \mathcal{C}$$

is s.m.

Example

Consider $(\mathcal{S}, \times)$.

$\mathrm{Alg}(\mathcal{S})$ the cat of "monoids in $\mathcal{S}$"

We have $\pi_0(X)$ is a monoid in Set

We call X grouplike if $\pi_0(X)$ is a group. ($\iff X \times X \xrightarrow{\sim} X \times X$, $(a,b) \longmapsto (ab, b)$)

An $\mathbb{E}_n$-Alg X in $\mathcal{S}$ is called grouplike if the underlying $\mathbb{E}_1$-Alg is.

For any $X \in \mathcal{S}_*$.

$$\Omega^n X \simeq \mathrm{Map}_{\mathcal{S}_*}(S^n, X) \cong \mathrm{Map}((D^n, \partial D^n), (X, *))$$

is canonically an $\mathbb{E}_n$-Alg in $(\mathcal{S}, \times)$.

Thm (Boardman – Vogt) $0 \le n < \infty$.

$$\Omega^n : \mathcal{S}_*^{n\text{-conn.}} \longrightarrow \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S})$$

is fully faithful w/ essential image

grouplike $\mathbb{E}_n$-Alg

Cor $A \in \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S})$,

$X \in \mathcal{S}_*^{\text{conn.}}$.

We have

$$\mathrm{Map}_{\mathbb{E}_n}(\underbrace{\Omega^n \Sigma^n X}_{\text{"free stuff generated by } X\text{"}}, A) \simeq \mathrm{Map}_{\mathcal{S}_*}(X, A)$$

Pf Sketch WLOG, A is connected and so grouplike. $A \simeq \Omega^n Y$.

$$\mathrm{Map}_{\mathbb{E}_n}(\Omega^n \Sigma^n X, A) \overset{\text{f.f.}}{\simeq} \mathrm{Map}_{\mathcal{S}_*}(\Sigma^n X, Y)$$
$$\simeq \mathrm{Map}_{\mathcal{S}_*}(X, \Omega^n Y) \quad \cdots$$

Pf Sketch of B-V thm

$n=1$,

$$\mathcal{S}_*^{\text{conn.}} \underset{\text{Bar}}{\overset{\Omega}{\rightleftarrows}} \mathrm{Alg}^{gr}_{\mathbb{E}_1}(\mathcal{S})$$

Second step

$$\mathcal{S}_*^{2\text{-conn}} \rightleftarrows \mathrm{Alg}^{gp}_{\mathbb{E}_1}(\mathcal{S}_*^{\text{conn}}) \rightleftarrows \mathrm{Alg}^{gp}_{\mathbb{E}_1}\mathrm{Alg}^{gp}_{\mathbb{E}_1}(\mathcal{S})$$

Thm (Dunn-Additivity)

$\mathcal{C}$ s.m. ∞-cat

$$\mathrm{Alg}_{\mathbb{E}_{m+n}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_m}\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}).$$

$(0 \le n, m \le \infty)$.

<u>Cor</u> $\mathcal{C}$ has a geometrical realization and $\otimes$ commutes w/ them in both variables.

$$\mathrm{Bar} : \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})_{/1} \longrightarrow \mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathcal{C}).$$

$$A \longmapsto \mathrm{colim}\left(\cdots A\otimes A \mathrel{\substack{\rightarrow\\ \rightarrow\\ \rightarrow}} A \rightrightarrows 1\right)$$