Cotangent Complex and Obstruction Theory

<u>sifted cat</u> are 1-cat $\mathcal{D}$ s.t.

any $\mathcal{D} \to \mathrm{Set}$ commutes w/ finite

product.

<u>Prop</u>. For $\mathcal{D}$ inhabited small,

it's sifted iff $\mathcal{D} \xrightarrow{\Delta} \mathcal{D} \times \mathcal{D}$

is final.

<u>Prop</u>. If $\mathrm{cospan}_{\mathcal{D}}(d_1, d_2)$ is connected

for any $d_1, d_2 \in \mathcal{D}$, then $\mathcal{D}$ is sifted.

<u>Prop</u> Δ^{op} is sifted.

<u>sifted ∞-cat</u> := nonempty ∞-cat (quasicat)

K s.t. $\Delta : K \to K \times K$ models a

final ∞-functor.

<u>Prop</u>. Given ∞-Cat $\mathcal{C}$ s.t. products preserve

sifted ∞-colimits (e.g. ∞-topos), then sifted colimits preserves finite products.

Recall $\mathbb{L}\Omega^n_{-/k} : \mathrm{Anim}(k\mathrm{Alg}) \longrightarrow \mathcal{D}(\mathbb{Z})_{\geq 0}$

I think it should be $\mathcal{D}(k)_{\geq 0}$.

Lemma These two agree:

(1) $\mathbb{L}\Omega^n_{-/k} : \mathrm{Anim}(k\mathrm{Alg}) \to \mathcal{D}(\mathbb{Z})_{\geq 0}$ evaluated on R

(2) $\mathbb{L}(A \mapsto R \otimes_A \Omega^n_{A/k}) : \mathrm{Anim}(k\mathrm{Alg}) \to \mathcal{D}(k)_{\geq 0}$ evaluated on R

Proof: resolve $R = \operatorname{colim}_{\Delta^{op}} A_i$, A_i poly.

$$(2) \simeq \operatorname{colim}_{\Delta^{op}} R \otimes_{A_i} \Omega^n_{A_i/k}$$

$$\simeq \operatorname{colim}_{\Delta^{op}} \left(\operatorname{colim}_{\Delta^{op}} A_j \right) \otimes^{\mathbb{L}}_{A_i} \Omega^n_{A_i/k}$$

$$\simeq \operatorname{colim}_{\Delta^{op} \times \Delta^{op}} A_j \otimes^{\mathbb{L}}_{A_i} \Omega^n_{A_i/k}$$

$$\simeq \operatorname*{colim}_{\Delta^{op}} A_i \otimes^{\mathbb{L}}_{A_i} \Omega^{n}_{A_i/k} \quad (\Delta^{op} \text{ sifted})$$

$$\simeq \operatorname*{colim}_{\Delta^{op}} \Omega^{n}_{A_i/k}$$

$$\simeq (1).$$

Ob. This $\mathrm{Anim}(k\text{-}\mathrm{Alg}_{/R}) \longrightarrow \mathcal{D}(R)_{\geq 0}$ commutes w/ colimits.

Example Take $R = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$ being regular seq.

$$k[y_1, \dots, y_m] \xrightarrow{y_i \mapsto f_i(x_1, \dots, x_n)} k[x_1, \dots, x_n]$$

then

$$\begin{array}{ccc} k[f_1, \dots, f_m] & \longrightarrow & k[x_1, \dots, x_n] \\ \downarrow & & \downarrow \\ k & \longrightarrow & R \end{array}$$

(is trivially a pushout diagram in $k\mathrm{Alg}$)

is a pushout in $\mathrm{Anim}(k\text{-}\mathrm{Alg}_{/R})$.

(using regularity condition)

Apply $\mathbb{L}(R \otimes_{\cdot} \Omega^1_{\cdot/k})$ yields

$$\begin{tikzcd}
R\{dy_1, \dots, dy_m\} \arrow[r, "(df_1, \dots, df_m)"] & R\{dx_1, \dots, dx_n\} \\[-2em]
R\{df_1, \dots, df_m\} \arrow[r, "\text{Jacobi matrix}"'] \arrow[d] & R\{dx_1, \dots, dx_n\} \arrow[d] \\
0 \arrow[r] & \mathbb{L}\Omega^1_{R/k} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

in $D(R)_{\geq 0}$.

We have $\mathbb{L}\Omega^1_{R/k}$ has $H_0 = \operatorname{Coker}(\text{Jacobi})$,

$H_1 = \ker(\text{Jacobi})$, other $H_* = 0$.

as $R\{dx_1, \dots, dx_n\}$ & $R\{dy_1, \dots, dy_m\}$ concentrate in deg 0.

Def <u>Cotangent Complex</u> of R/k

$L_{R/k} := \mathbb{L}\Omega^1_{R/k}$.

Split square zero extension

$S \in k\mathrm{Alg}_{\text{com.}}$, $M \in \mathrm{Mod}_S$.

M has a split square zero extension

$$S \oplus M \in k\mathrm{Alg}$$

that is, we force multiplication on M vanish.

Ob.

$$\begin{tikzcd} & S\oplus M \arrow[d] \\ R \arrow[r, "\varphi"'] \arrow[ur, dashed, "\text{lift}"] & S \end{tikzcd} \rightsquigarrow \varphi\text{-linear derivation } R \to M,$$

i.e.

R-mod maps

$$\Omega^1_{R/k} \to M_\varphi$$

Prop. - We get a derived functor
("derived ver" of Ob above)

$$\mathcal{D}(S)_{\geq 0} \longrightarrow \mathrm{Anim}(k\text{-Alg})_{/S}$$

on projectives: $P \mapsto S \oplus P$.

- we have equiv's

$M \in \mathcal{D}(S)_{\geq 0}$

$$\mathrm{Map}_{\mathcal{D}(R)_{\geq 0}}(L_{R/k}, M_\varphi) \simeq \mathrm{fib}_\varphi \left(\begin{array}{c} \mathrm{Map}_{\mathrm{Anim}(k\mathrm{Alg})}(R, S \oplus M) \\ \downarrow \\ \mathrm{Map}_{\mathrm{Anim}(k\mathrm{Alg})}(R, S) \end{array} \right)$$

fiber over φ

Given a pair $\tilde{R} \twoheadrightarrow R$, kernel I satisfies $I^2 = 0$.

Note: I can be made an R-mod naturally, as $I^2 = 0$.

This pair is called a (nonsplit) square zero extension.

Then $L_{R/\tilde{R}}$ has $H_0 = 0$, and

$$H_1 = R \otimes_{\tilde{R}} I = I.$$

$$\rightsquigarrow \quad L_{R/\tilde{R}} \longrightarrow I[1] \quad \text{(of } R\text{-mods)}$$

(iso on H_1)

$$\rightsquigarrow \quad R \xrightarrow{\delta} R \oplus I[1] \quad \text{(of animated } \tilde{R}\text{-algs)}$$

Prop We have a pullback in $\mathrm{Anim}(\tilde{R}\,\mathrm{Alg})$

$$\begin{tikzcd} \tilde{R} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & R \arrow[d, "s"] \\ R \arrow[r, "\delta"] & R \oplus I[1] \end{tikzcd}$$

Prop (unproved)

(non split) square zero (k-alg) extension

$$\tilde{R} \to R \text{ along } I$$

corresponds to $\underline{L_{R/k}} \to I[1]$.

(indeed for $f: A \to B$, have $f_! L_A \to L_B \to L_{B/A}$.)

Eg. $R \xrightarrow{s} R \oplus I[1]$

corresponds to the null map

$$L_{R/k} \to I[1]$$

the pullback square above says that

$$\text{lifts} \quad \begin{tikzcd} & \tilde{R} \arrow[d] \\ S \arrow[r] \arrow[ur, dashed] & R \end{tikzcd} \longleftrightarrow \begin{tikzcd} & R \arrow[d, "s"] \\ S \arrow[r, "\delta"] \arrow[ur, dashed] & R \oplus I[1] \arrow[u, bend right, "pr"'] \end{tikzcd}$$

$\longleftrightarrow$ homotopy $S \to R \to R \oplus I[1]$

$\simeq S \xrightarrow{\delta} R \oplus I[1]$

$\longleftrightarrow$ nullhomotopy of $L_{S/k} \to I[1]$

These null homotopies form a <u>torsor</u> over $\pi_1 \mathrm{Map}(L_{S/k}, I[1])$

Summary:

- $\tilde{R}$ are classified by

$\pi_0 \mathrm{Map}_{\mathcal{D}(R)}(L_{R/k}, I[1])$

- For $S \to R$, there is an obstruction in $\pi_0 \mathrm{Map}_{\mathcal{D}(S)}(L_{S/k}, I[1])$ for existence of lifts $S \to \tilde{R}$.

- lifts (if exists) are parametrized by

$$\pi_1 \mathrm{Map}_{\mathcal{D}(S)}(L_{S/k}, I[1])$$

$$\simeq \pi_0 \mathrm{Map}_{\mathcal{D}(S)}(L_{S/k}, I).$$

Exercise

$S \in k\mathrm{Alg}$.

TFAE.

- For every square-zero ext $\tilde{R} \to R$, and every $S \to R$, there is a lift
- $H_1 L_{S/k} = 0$, $H_0 L_{S/k}$ is a projective S-module..
- formally smooth

Thm If $R_1 \in \mathbb{F}_p\mathrm{Alg}$ is perfect, then there exists flat $\mathbb{Z}/p^n$-alg R_n unique up to iso, s.t.

$$R_1 \cong R_n \otimes_{\mathbb{Z}/p^n} \mathbb{F}_p$$

In particular, $R_{n-1} \cong R_n \otimes_{\mathbb{Z}/p^n} \mathbb{Z}/p^{n-1}$.

$$R := \lim \begin{pmatrix} \cdots \\ \downarrow \\ R_n \\ \downarrow \\ R_{n-1} \\ \downarrow \\ \cdots \end{pmatrix}$$

called (p-typical) Witt vectors of R_0. denoted by $W(R_0)$

is the unique flat, p-complete $\mathbb{Z}_p$-alg

w/ $R_1 \cong R \otimes_{\mathbb{Z}_p} \mathbb{F}_p = R/p$.

R_n = square-zero extension of R_{n-1} by R_1 : indeed,

$$\left(0 \to \mathbb{Z}/p \to \mathbb{Z}/p^n \to \mathbb{Z}/p^{n-1} \to 0\right)$$

$R_n \otimes_{\mathbb{Z}/p^n}$

$$= \quad 0 \to \underset{\substack{\Uparrow \\ \text{square zero ideal}}}{R_1} \to R_n \to R_{n-1} \to 0$$

So it's classified by

$$\pi_0 \mathrm{Map}_{\mathcal{D}(R_{n-1})}(L_{R_{n-1}/\mathbb{Z}}, R_1[1])$$

and compatible w/ the corresponding class

in $\pi_0 \mathrm{Map}_{\mathcal{D}(\mathbb{Z}/p^{n-1})}(L_{(\mathbb{Z}/p^{n-1})/\mathbb{Z}}, \mathbb{Z}/p[1])$

that classifying $\mathbb{Z}/p^n \to \mathbb{Z}/p^{n-1}$.

<u>Lem</u> Iso classes of such R_n are

a torsor over $\pi_0 \mathrm{Map}_{\mathcal{D}(R_{n-1})}(L_{R_{n-1}/\mathbb{Z}/p^{n-1}}, R_1[1])$

$$\simeq \pi_0 \mathrm{Map}_{\mathcal{D}(R_1)}(\underset{\sim}{L_{R_1/\mathbb{F}_p}}, R_1[1]).$$

indeed $\simeq 0$

followed from

R_1 being perfect

just consider

$$(-)^p : \Omega^1_{R_1/\mathbb{F}_p} \xrightarrow{0} \Omega^1_{R_1/\mathbb{F}_p}$$

$$\rightsquigarrow \underset{0 = \text{iso}}{\circlearrowright} L_{R_1/\mathbb{F}_p}$$

$$\rightsquigarrow L_{R_1/\mathbb{F}_p} \simeq 0.$$

<u>Example</u> $\mathbb{F}_{p^n}/\mathbb{F}_p$ lifts to $\mathbb{Z}_p$-alg

$W(\mathbb{F}_{p^n})$, w/ $W(\mathbb{F}_{p^n})/p \simeq \mathbb{F}_{p^n}$.

e.g. write $\mathbb{F}_p[x]/(f(x))$,

take $\mathbb{Z}_p[x]/(\underline{\tilde{f}}(x))$

some lifts of f.