Connes operator

Def R assoc. k-alg.

We define a k-linear map

$$B : HH(R/k)_n \longrightarrow HH(R/k)_{n+1}$$

$$r_0 \otimes \cdots \otimes r_n \longmapsto$$

$$\sum_{\sigma \in C_{n+1}} (-1)^{n\sigma(0)} \cdot (1 \otimes r_\sigma - (-1)^n r_\sigma \otimes 1)$$

$$r_\sigma := r_{\sigma(0)} \otimes \cdots \otimes r_{\sigma(n)}$$

C_{n+1} = group of cyclic permutations of $\{0, \dots, n\}$

Ex B satisfies

- $B^2 = 0$
- $dB + Bd = 0$

B equips $HH(R/k)$ w/ the struct. of a DG-module over the DGA

$$A = \mathbb{Z}[b]/b^2, \quad |b| = 1, \quad \partial = 0$$

$$= H_*(\Pi, \mathbb{Z})$$

$\Rightarrow$ $HH_*(R/k)$ is a graded module over $\mathbb{Z}[b]/b^2$.

In particular, have an operator

$B: HH_*(R/k) \longrightarrow HH_{*+1}(R/k)$

w/ $B^2 = 0$.

Fact: If R is com, then B is a derivation.

It's not true on $HH(\mathbb{F}_p/k)$.

Prop. $\Omega^*_{R/k} \longrightarrow HH_*(R/k)$

sends d to B.

Example $HH_*(\mathbb{F}_p/\mathbb{Z}) \simeq \mathbb{F}_p\langle x\rangle$, $|x|=2$.

B is trivial (for degree reasons).

Q B acts trivially on $HH(\mathbb{F}_p/\mathbb{Z})$?

Defn. (derived) mod p reduction

$$HH(\mathbb{F}_p)/\!/p := HH(\mathbb{F}_p) \otimes^L_{\mathbb{Z}} \mathbb{F}_p$$

$$\simeq HH(\mathbb{F}_p) \otimes_{\mathbb{Z}}$$

$$(\Lambda_{\mathbb{Z}}(\varepsilon),\ |\varepsilon| = 1,\ \partial\varepsilon = p)$$

$$\simeq HH(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(\varepsilon)$$

$(\partial\varepsilon = 0,$

$$\Rightarrow \quad H_*(\cdots /\!\!/ p) \cong HH_*(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(\varepsilon)$$

$$\cong \mathbb{F}_p\langle x\rangle \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(\varepsilon)$$

$$|\varepsilon| = 1, \quad |x| = 2.$$

Prop. We have

$$B(x^{[n]}) = 0, \quad B(\varepsilon) = x.$$