Bökstedt Periodicity

__Thm__ (Bökstedt ver. 2)

$THH(\mathbb{F}_p)$ is free on 1 deg 2 generator as $H\mathbb{F}_p - \mathbb{E}_1 -$ alg.

$$THH(\mathbb{F}_p) \simeq H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3$$

$\Omega\Sigma S^2$ is the free $\mathbb{E}_1$-alg on pointed S^2,

$$\mathrm{Map}_{\mathbb{E}_1\text{-}\mathrm{Alg}/H\mathbb{F}_p}(H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3, R)$$

$$\simeq \mathrm{Map}_{\mathbb{E}_1\text{-}\mathrm{Alg}}(\Omega S^3, \Omega^\infty R)$$

$$\simeq \mathrm{Map}_{S_*}(S^2, \Omega^\infty R)$$

$$\simeq \mathrm{Map}_{Sp}(\Sigma^\infty S^2, R)$$

$$\simeq \mathrm{Map}_{H\mathbb{F}_p}(\Sigma^2 H\mathbb{F}_p, R)$$

Proof of equivalence

$x \in THH_2(\mathbb{F}_p)$

$\rightsquigarrow$ $\mathbb{E}_1$-map $\quad H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3 \longrightarrow THH(\mathbb{F}_p)$

which is $\simeq$ iff $\quad THH_*(\mathbb{F}_p) = \mathbb{F}_p[x]$.

$\pi_*(\text{left}) \cong H_*(\Omega S^3; \mathbb{F}_p)$ $\leftsquigarrow$ $\mathbb{E}_1$-alg struct on ΩS^3 gives ring struct. on H_*.

$\cong \mathbb{F}_p[x]$

Lem $\quad THH(R) \simeq R \underset{R \otimes_{\mathbb{S}} R^{op}}{\otimes} R$

Sketch $\quad R \simeq R \otimes_R R = \left| R \otimes_{\mathbb{S}} R \leftleftarrows R \otimes_{\mathbb{S}} R \otimes_{\mathbb{S}} R \leftleftarrows \cdots \right|$

(R-bimods)

$$R \underset{R \otimes_{\mathbb{S}} R^{op}}{\otimes} R = \left| R \underset{R \otimes_{\mathbb{S}} R^{op}}{\otimes} \left(\leftleftarrows \cdots \leftleftarrows \cdots \right) \right|$$

$= |\text{cyclic bar construction}|$

$= THH(R)$

Need to understand $H\mathbb{F}_p \underset{\mathbb{S}}{\otimes} H\mathbb{F}_p$

dual steenrod alg.

$\mathcal{A}_*$

<u>Thm</u> (Milnor)

$\pi_*(H\mathbb{F}_p \underset{\mathbb{S}}{\otimes} H\mathbb{F}_p)$ is :

case $p=2$: $\mathbb{F}_2[\zeta_1, \zeta_2, \cdots]$, $|\zeta_i| = 2^i - 1$.

p odd : $\Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, \cdots) \otimes \mathbb{F}_p[\xi_1, \xi_2, \cdots]$

$|\tau_i| = 2p^i - 1$

$|\xi_i| = 2p^i - 2$.

View $H\mathbb{F}_p \underset{\mathbb{S}}{\otimes} H\mathbb{F}_p$ as $H\mathbb{F}_p$-($\mathbb{E}_\infty$)-alg via inclusion of 2nd $H\mathbb{F}_p$.

Lem As an $\mathbb{E}_2$-$H\mathbb{F}_p$-alg,

$H\mathbb{F}_p \otimes_{\mathbb{S}} H\mathbb{F}_p$ is free on one generator of deg 1, i.e.

$$H\mathbb{F}_p \otimes \underline{\Sigma^\infty_+ \Omega^2 S^3} \xrightarrow{\sim} H\mathbb{F}_p \otimes_{\mathbb{S}} H\mathbb{F}_p.$$

is really just $\mathbb{E}_2$ (or $\mathbb{E}_3$)-alg; only free as $\mathbb{E}_2$-alg.

Lem $\overset{+\cdots}{\Longrightarrow}$ Bökstedt:

$$THH(\mathbb{F}_p) = H\mathbb{F}_p \underset{H\mathbb{F}_p \otimes_{\mathbb{S}} H\mathbb{F}_p^{op}}{\otimes} H\mathbb{F}_p$$

$$\overset{\text{Lem}}{=} H\mathbb{F}_p \underset{H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega^2 S^3}{\otimes} H\mathbb{F}_p$$

$$\simeq H\mathbb{F}_p \otimes \Sigma^\infty_+ \underbrace{\mathrm{Bar}(*, \Omega S^3, *)}_{\text{deloop}}$$

$$\simeq H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega S^3$$

$\mathrm{Bar}(*, \mathbb{E}_1\text{-alg}, *)$

$=$ deloop.

Connective space = group-like $\mathbb{E}_1$-alg

May recognition thm, [HA, Thm 6.2-6.15]

$\mathcal{H}$ ∞-topos (e.g. $\mathcal{S} = \infty\mathrm{Grpd}$)

then have

$$\underbrace{\mathrm{Groups}_{\mathbb{E}_n}(\mathcal{H})}_{\mathbb{E}_n\text{-groups}} \underset{B^n}{\overset{\Omega^n}{\rightleftarrows}} \underbrace{\mathcal{H}^{*/}_{\geq n}}_{\text{pointed } (n-1)\text{-connected objects}}$$

Rmk Indeed Len $\Leftrightarrow$ Bökstedt.

using: $A \to B$ of connected $H\mathbb{F}_p$-alg

no negative htpy groups

is $\sim$ iff

$$H\mathbb{F}_p \otimes_A H\mathbb{F}_p \xrightarrow{\sim} H\mathbb{F}_p \otimes_B H\mathbb{F}_p$$

applied to $H\mathbb{F}_p \underset{\mathbb{S}}{\otimes} \Sigma^\infty_+ \Omega^2 S^3 \longrightarrow H\mathbb{F}_p \underset{\mathbb{S}}{\otimes} H\mathbb{F}_p$

<u>Ex</u> - A connected $H\mathbb{F}_p$-alg.

Then $N \to M$ of connected A-mods

is $\sim$ iff $H\mathbb{F}_p \underset{A}{\otimes} N \xrightarrow{\sim} H\mathbb{F}_p \underset{A}{\otimes} M$.

- $A \to B$ connected $H\mathbb{F}_p$-alg.

$H\mathbb{F}_p \underset{A}{\otimes} H\mathbb{F}_p \xrightarrow{\sim} H\mathbb{F}_p \underset{B}{\otimes} H\mathbb{F}_p$.

then $H\mathbb{F}_p \otimes_A B \simeq H\mathbb{F}_p$,

$A \xrightarrow{\sim} B$.

p - odd : for $\mathbb{E}_\infty - H\mathbb{F}_p$ - alg. have

Dyer - Lashof operations.

introduced in a paper of iterated loop spaces

$i \in \mathbb{Z}$, $Q^i : \pi_n(R) \longrightarrow \pi_{n+2(p-1)i}(R)$.

$\beta Q^i : \pi_n(R) \longrightarrow \pi_{n+2(p-1)i-1}(R)$.

Somehow coming from homology of Confs.

they satisfy :

- $|x| = n$ even, $Q^{\frac{n}{2}} x = x^p$

$$Q^i x = 0 \ , \ i < \frac{n}{2}$$

$$(\beta Q^i x = 0 , \ i \leq \frac{n}{2})$$

For $\mathbb{E}_2$-alg,

nothing new $\longrightarrow$ $Q^{\frac{n}{2}} x$ for n-even,

$Q^{\frac{n+1}{2}} x , \ \beta Q^{\frac{n+1}{2}} x$ for n odd

new

are already defined

Thm (Dyer – Lashof, $p=2$ Araki–Kudo)

$$H_*(\Omega^2 S^3; \mathbb{F}_p) = \Lambda(\underset{\tau_0}{a}, \underset{\tau_1}{Q^1 a}, \underset{\tau_2}{Q^p Q^1 a}, \cdots)$$

$$\otimes \mathbb{F}_p[\underset{\xi_1}{\beta Q^1 a}, \underset{\xi_2}{\beta Q^p Q^1 a}, \cdots]$$

$|a| = 1$

Recall:
$\Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, \cdots) \otimes \mathbb{F}_p[\xi_1, \xi_2, \cdots]$
$|\tau_i| = 2p^i - 1$
$|\xi_i| = 2p^i - 2$.

Thm (Steinberger)

On $\pi_*(H\mathbb{F}_p \otimes_{\mathbb{S}} H\mathbb{F}_p)$, have

$$\tau_i = Q^{p^{i-1}} Q^{p^{i-2}} \cdots Q^1 \tau_0$$

$$\xi_i = \beta Q^{p^{i-1}} Q^{p^{i-2}} \cdots Q^1 \tau_0$$

Proof of Lem.

Recall: Lem As an $\mathbb{E}_2$-$H\mathbb{F}_p$-alg, $H\mathbb{F}_p \otimes_{\mathbb{S}} H\mathbb{F}_p$ is free on one generator of deg 1, i.e.

$$H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega^2 S^3 \xrightarrow{\sim} H\mathbb{F}_p \otimes_{\mathbb{S}} H\mathbb{F}_p.$$

$$\pi_1(H\mathbb{F}_p \otimes_{\mathbb{S}} H\mathbb{F}_p) \xrightarrow{\cong} \pi_1(H\mathbb{F}_p \otimes_{H\mathbb{Z}} H\mathbb{F}_p)$$

$\mathbb{S} \to H\mathbb{F}_p$ is 1-connected

$$\cong \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{F}_p, \mathbb{F}_p) \cong \mathbb{F}_p$$

get an $\mathbb{E}_2$-$H\mathbb{F}_p$-alg map

$$H\mathbb{F}_p \otimes \Sigma^\infty_+ \Omega^2 S^3 \longrightarrow H\mathbb{F}_p \otimes_{\mathbb{S}} H\mathbb{F}_p$$

iso on π_1. Both sides are generated in the same way by $\mathbb{E}_2$-Dyer-Lashof operations.

So iso.