Vesna Stojanoska

Introduction to Chromatic Homotopy Theory

Beginnings

Quillen

MU $\longleftrightarrow$ formal grp laws

$MU \xrightarrow[\text{complex orient}]{} E$

BP Peterson Sp

Adams-Novikov SS

in char p

classified by height

$0 \leq h \leq \infty$

height filtration

$[p]_F(x) = v_h x^{p^h} + \text{higher}$

unit

chromatic SS (Miller-Ravenel-Wilson)

Has homotopical origins

$K(n)$ Marava E-theory

$$K(n)_* \cong \mathbb{F}_p[v_n^{\pm 1}] \qquad n \geq 1$$

with a fgl (Honda)

$$[p]_{F_n}(x) = v_n x^{p^n}$$

eg $v_0 = p$, $K(0) = H\mathbb{Q}$

$$K(1) = L/p$$

Adams L-summand of KU

$S_p \rightsquigarrow$ localize $L_{K(n)}$

$$L_n = L_{K(0)\vee\cdots\vee K(n)}$$

- Chromatic convergence (Hopkins – Ravenel)

$$S^0_{(p)} \simeq \lim_n L_n S^0$$

- Fracture squares

$$\begin{tikzcd}
L_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{K(n)} \arrow[d] \\
L_{n-1} \arrow[r] & L_{n-1}L_{K(n)} \arrow[l, dashed, bend right]
\end{tikzcd}$$

(+ chromatic splitting conjecture)

$\rightsquigarrow$ $K(n)$-local homotopy theory

Formal grp laws have nice deformations

(Lubin - Tate)

(F, k) $\rightsquigarrow$ p-complete ring carrying a universal deformation of (F, k)

fgl over

(char $k = p$

k perfect)

Goerss-Hopkins-Miller

$E(F, k)$ E_∞-ring spectrum $K(hHF)$-local

Morava E-theory

functoriality $\rightsquigarrow$ $\mathrm{Aut}(F, k)$ acts on $E(F, k)$ via E_∞-ring maps

nice profinite grp

Notation $E_n = E(F_n, \mathbb{F}_{p^n})$; $\mathbb{G}_n = \mathrm{Aut}(F_n, \mathbb{F}_{p^n})$

Morava stabilizer grp

e.g. $E_1 = KU_p^\wedge$, $\mathbb{G}_1 = \mathbb{Z}_p^\times$ acts via

Adams operations

e.g. E_2 related to top. mod. forms

In general, $\pi_* E_n \cong (\pi_0 E)[u^{\pm 1}]$ $|u| = -2$

Lubin-Tate ring

$\pi_0 E \cong \mathbb{Z}_{p^n}[[u_1, \dots, u_{n-1}]]$

Thm (Devinatz-Hopkins) $S^0_{K(n)} \simeq E_n^{h\mathbb{G}_n}$

Morava

Rognes: $S^0_{K(n)} \longrightarrow E_n$ is a profinite Galois extension

(w/ Gal grp $\mathbb{G}_n$)

Baker-Richter: E_n is (almost) the alg. closure of $S^0_{K(n)}$

(need $E(\mathbb{F}_{p^n}, \overline{\mathbb{F}_p})$)

$E(F, \overline{k})$ is alg. closed in sense of chromatic Nullstellensatz (Burklund - Schlank - Yuan)

$X \mapsto (E_n \wedge X)$

In fact $\quad Sp_{K(n)} \simeq \left(\mathrm{Mod}_{K(n)}(E_n)\right)^{hG_n}$

$\mapsto {}^{hG_n}$ $\qquad$ $K(n)$-local E_n-modules

descent $\quad$ Mathew, Mor, ...

How to understand $(\ )^{hG_n}$?

$\underline{n=1}$ $\quad G_1 = \mathbb{Z}_p^\times$

p odd:

$$S^0_{K(1)} = E_1^{h\mathbb{Z}_p^\times}$$

$$\rightsquigarrow \quad S^0_{K(1)} \to KU_p^\wedge \xrightarrow{\psi^l - 1} KU_p^\wedge$$

(Adams-Baird-Ravenel, Bousfield)

$C_2 = \{\pm 1\} \subset \mathbb{Z}_2^\times$,

$p=2$

$$S^0_{K(1)} \to (E_1)^{hC_2} \to E_1^{hC_2}$$

$n=2$ Goerss-Henn-Mahowald-Rezk resolution

$(p \geq 3)$

$$S^0_{K(2)} \to \underbrace{E_2^{hF} \to \cdots}_{\text{length } 4 = 2^2}$$

finite subgrps of $\mathbb{G}_2$

vertical cohomo dim

$\Downarrow$

$\mathbb{G}_n$ vcd $= n^2$ $\longrightarrow$ finite length n^2 resolution in terms of

the finite subgrps $F \leq \mathbb{G}_n$

(i.e. $S^0_{K(n)}$ res in terms of E_n^{hF})

$\longrightarrow$ study E_n^{hF}

Finite p-subgrps are C_{p^i} or

($p=2$) Q_8

$\underline{p=2}$ $\{\pm 1\} = C_2 \subset \mathbb{G}_n$

(Hahn-Shi):

$$\begin{tikzcd} MU_{\mathbb{R}} \arrow[r, "C_2"] & E_n \arrow[d, squiggly] \\ C_2\text{-}Sp \arrow[u, squiggly] & \end{tikzcd}$$

orientation

E_n Borel $\rightsquigarrow$ genuine F-spectrum

E_∞-ring Sp $\rightsquigarrow$ "genuine E_∞" or N_∞

Blumberg Hill

$\rightsquigarrow \quad C_2 \subseteq F$

$$N^F_{C_2} MU_{\mathbb{R}} \xrightarrow{F} E_n$$

HHR norm