Mona Merling

G - manifolds & equivariant homotopy theory

Q Given a finite CW complex, is it homotopic to a compact manifold?

Obstruction: Poincaré duality

Equivariantly:

A G-CW cplx: equivariant pts = orbits

built out of cells $G/H \times D^n$

$H_*^{\mathrm{Borel}}(G/H, A)_{\text{ab gp}} \cong H_*^{\mathrm{op}\ \mathrm{hom}}(\Lambda$

same for H^*_{Borel}

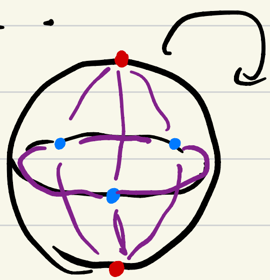
$\mathbb{Z}$-graded Bredon cohomology

$H^*_{a}(X, M)$

coeff. system

PC holds for $G/H \times M^{?}$ triv.

Ex. C_3



3 0-cells

2 - of type $C_3/C_3 \times D^0$

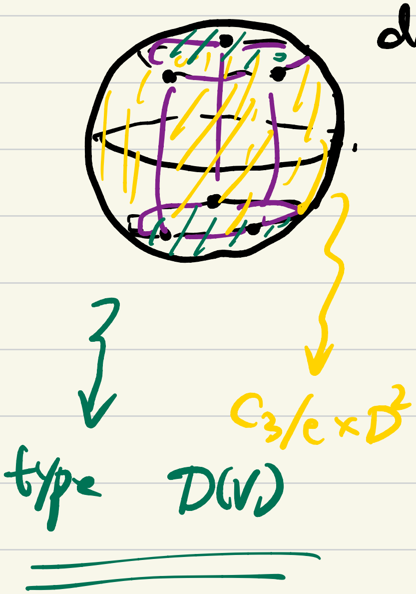
1 - of type $C_3/e \times D^0$

3 1-cells

of type $C_3/e \times D^1$

2 2-cells of type $C_3/e \times D^2$

dual



dual cells

2 0-cells of type $C_3/e \times D^0$

3 1-cells of type $C_3/e \times D^1$

Related to $H_a(\ ; M)$ not fully stable equivariantly

$$H_a^{n+1}(\Sigma X ; M) \cong H_a^n(X ; M)$$

but not w.r.t. Σ^V.

$\rightsquigarrow$ need to consider $RO(G)$-graded Bredon (co)homology

$\leftrightarrow$ represented by genuine G-spectra

→ Costenoble - Waner

☆ " Guide to parametrized cohomology"

Beaudry - Lewis - May - Pauli - Tatum

☆ " Parametrized Poincaré duality"

Hilman - Kirstein - Kremer

Q M, N smth cpt n-dim mflds

w/ $M \simeq N$, is $M \cong N$?

~~Strategy~~ · construct a h-cob W b/w

M & N: an $(n+1)$-dim cpt

mfld W $\partial W = M \sqcup N$

$M \hookrightarrow W$
$N \hookrightarrow W$ are $\simeq$

obstructions to h-cobordisms being trivial $\longrightarrow$ live in a group

<u>Thm</u> (Smale, Barden, Mazur, Stallings)

(h-cob thm / s-cob thm)

dim $M \geq 5$

$$\left\{ \begin{array}{c} \text{iso classes of h-cob} \\ \text{on } M \end{array} \right\} \longleftrightarrow Wh(\pi_1 M)$$

$$Wh(\pi_1 M) = K_1(\mathbb{Z}\pi_1 M) \big/ \{\pm g \in \pi_1 M\}$$

$$M \xrightarrow{\simeq} W \longleftrightarrow \tau(M \hookrightarrow W) = 0$$

Wh torsion

$$\Longleftrightarrow W \cong M \times I$$

Equivariantly

M^H — fixed pts

M_H — H-stratum $= \{ m \in M \mid \mathrm{Stab}_m = H \}$

$= M^H \setminus M^{>H}$

W_H Weyl group

$W_H \curvearrowright M^H$

$W_H \curvearrowright M_H$ freely

Def An equiv. h-cob W s.t.

$M \hookrightarrow W \hookleftarrow N$ are

G-homotopy equiv.

<u>Thm</u> (Araki - Kawakubo)

"eqvt h-cob thm".

(W; M, N) eqvt h-cob

$\tau_G(M \hookrightarrow W) = 0 \in Wh_G(M)$

$\rightsquigarrow$ define by Illman

$\Updownarrow$

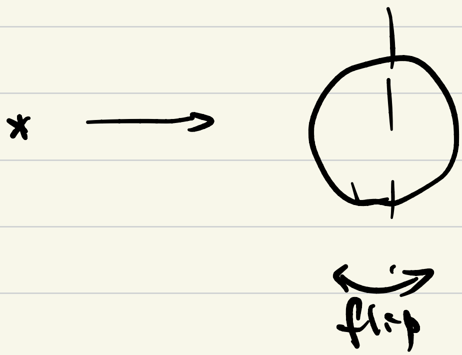
$\exists$ rep V s.t.

$W \times D(V) \cong M \times D(V) \times I$

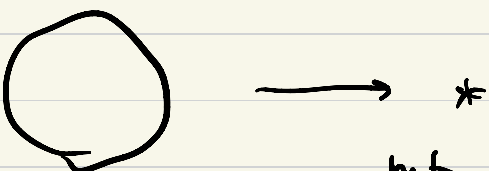
Def An eqvt map $X \xrightarrow{f} Y$ is isovariant if $G_x = G_{f(x)}$

(i.e. $X_H \subseteq f(Y_H)$)

?

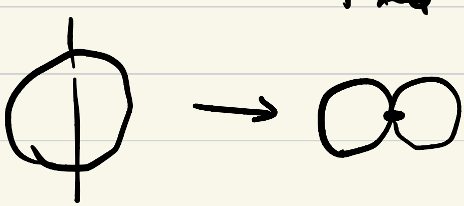


isov



not isov

(free points go to fixed pts)



isov.

Def. An isov. h-cob is s.t.

$$M \hookrightarrow W \hookleftarrow N \text{ are}$$

isov. h.e.

Thm $(W; M, N)$ isov. h-cob

(ref Lück   w/ $\dim M^H/_{WH} \geq 5$

Stemberger)

$$\tau_a^{isov}(M \hookrightarrow W) = 0$$

$$\Leftrightarrow W \cong M \times I.$$

Thm (Lück)

If M statisfies the weak gap hypotheses,

$(M^H \subsetneq M^K, \text{ then } \dim M^K \geq \dim M^H + 3)$

then W is eqvt. h-cob $\Leftrightarrow$ isov. h-cob.

Q: want to understand the homotopy type

$$\coprod_{[M]} B\mathrm{Diff}(M)$$

$\exists$ fiber seq.

$$\underbrace{\widetilde{\mathrm{Diff}}(M)/\mathrm{Diff}(M)}_{\substack{\text{computed in} \\ \text{a range by } \mathcal{H}^{\infty}(M)}} \to \underset{?}{B\mathrm{Diff}(M)} \to \underbrace{B\widetilde{\mathrm{Diff}}(M)}_{\substack{\text{accessible via} \\ \text{surgery theory}}}$$

block diffeos

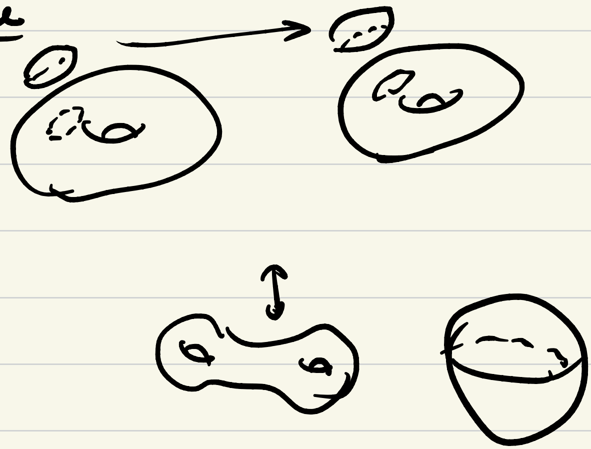
(Weiss-Williams, Hatcher) "stable h-cob "space" w/ $\pi_0 = Wh(\pi_1 M)$

Q: When are two cpt manifolds scissors congruent?

"SK" = "schneiden und kleben"

= "cut and paste"

Example



Def. $\mathcal{M}_n$ = monoid of diffeo classes of cpt n-dim mfld w/ $\amalg$

(Karras - Kreck - Neumann - Ossa)

$SK_n = Gr(\mathcal{M}_n) / \sim_{SK}$

$\rightsquigarrow$ computed completely

The only SK-invariant is χ

Euler char.

Thm (Hoekzema - M - Murray - Rovi - Semikina)

$\exists$ Sp $K^{\square}(\mathrm{Mfld}_n^{\partial})$

w/ $K_0 = SK_n^{\partial}$ &

$$K^{\square}(\mathrm{Mfld}_n^{\partial}) \longrightarrow K(\mathbb{Z})$$

on π_0 $SK_n^{\partial} \longrightarrow \mathbb{Z}$

$$[M] \longmapsto \chi(M).$$

Prop (M - Raptis - Semikina)

on π_1: $K_1^{\square}(\mathrm{Mfld}_n^{\partial}) \longrightarrow \mathbb{Z}/2\mathbb{Z}$

$[M, f] \longmapsto \chi_{1/2}(M)$

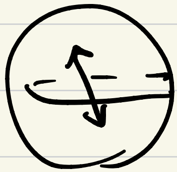
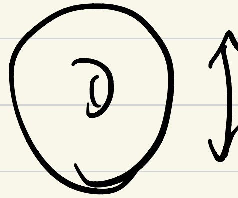
SKK-invariant

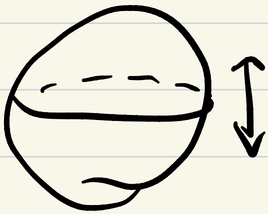
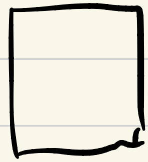
(Keep track of glueing diffeo.)

$SKK_n \cong \pi_0 MTSO$

$\cong \pi_1 B\mathrm{Cob}_n$

Equivariantly

Exercise  antipodal $\sqcup$  flip

$\underset{C_2\text{-}SK}{\sim}$  flip $\sqcup$  diag action

Def : $SK_n^{G,\partial} = Gr(\mathcal{M}_n^{G,\partial}) \big/ \sim SK_G^{\partial}$

X G-CW cplx

$\chi_a(X) = \sum_{k \geq 0} [\text{set of } k\text{-cells}] \in A(G)$

but not only invariant

Rmk. $SK_n^{G,\partial} \cong K_0^{\square}(\mathrm{Mfld}_n^{G,\partial})$

Thm (M-Ng-Semikina-(Sendon-Blanco)-Williams)

$$K_0^{M}(\mathrm{Mfld}_n^{G,\partial}) \to K_G(\underline{\mathbb{Z}})^G$$

defined Calle-Chan-Mejia

on π_0 recovers $\chi_G(M)$

Prop SK^H form a Mackey functor

Conj. this should be the fixed pt map of G-spectra.

<u>ref</u> noneqv.

"Lecture notes on the stable parametrized h-cob thm" - Rognes

eqvt. π_0-version

"Transformation groups & alg. K-theory"

- Lück

G-manifolds and equivariant K-theory

Recall

Thm (S, B, M, S)

s-cob / h-cob

dim $M \geq 5$, M cpt smth

$$\left\{ \begin{array}{l} \text{Iso classes of} \\ \text{h-cobs of } M \end{array} \right\} \longleftrightarrow Wh(\pi_1 M) = K_1(\mathbb{Z}\pi_1 M) \Big/ \pm g \in \pi_1 M$$

$\mathcal{H}(M)$ = space of h-cob on M up to diffeo +

$M \cup \partial M \times I$ & nbhd of

Q: Are all higher htpy gps also computed by K-theory?

A yes but only in a range.

Stab map $\mathcal{H}(M) \longrightarrow \mathcal{H}(M \times I)$

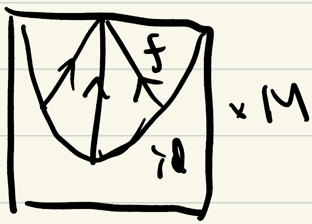
Interlude think about pseudoisotopies

$M \times I \xrightarrow[f]{\cong} M \times I$ id on $M \times 0 \amalg \partial M \times I$

Stab on map $\mathcal{P}(M) \longrightarrow \mathcal{P}(M \times I)$

$$M \times I \xrightarrow[f]{} M \times I \rightsquigarrow M \times I \times I \xrightarrow{?} M \times I \times I$$

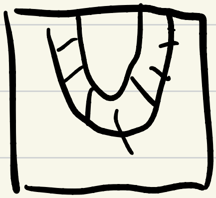
Idea



Problem need to put to condition to ensure smoothness at the cone pt

stab loses this

Idea 2 : U-shape



$P(M) \to P(M \times I) \to P(M \times I \times I)$

Best idea yet

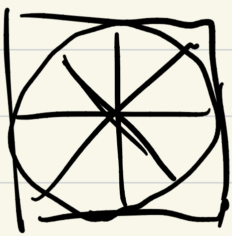
For f consider its double $\bar{f}$

$M \times [-1,1] \longrightarrow M \times [-1,1]$

Say f is mirror if $\bar{f}$ is smooth

can show $P^{\text{mirror}}(M) \simeq P(M)$.

$St^V f \quad M \times D(V) \times [-1,1] \to M \times D(W) \times [-1,1]$



Smooth!

$$St^V St^W = St^{V \oplus W}$$

<u>Idea</u>. use ~~this~~ for h-cob.

Smoothness is a bit of work!

$$St^V St^W \simeq St^{V \oplus W}$$

Good idea:

work w/ corners

Thm (Igusa)

$$\mathcal{H}(M) \longrightarrow \mathcal{H}^\infty(M) = \operatorname{colim}_k \mathcal{H}(M \times I^k)$$

$\sim n/3$ conn. $\quad n = \dim M$

recall. $\mathcal{H}^\infty(M)$ is a main ingredient for computing $\pi_* B\mathrm{Diff}(M)$

Thm (Waldhausen + Jahren, Rognes)

$$A(X) \simeq QX_+ \times Wh(X)$$

"alg K-theory of X"

w/ $\Omega Wh(M) \simeq \mathcal{H}^\infty(M)$ for M cpt smth mfld "h-cob/s-cob thm?"

$\rightsquigarrow \pi_0 H^\infty(M)$ K-group

$\mathcal{H}^\infty(M) \rightsquigarrow$ K-theory space

Goal / conjecture

<u>G - finite</u>

$M \circlearrowleft G$ smooth cpt

w/ "trivial G-corners"

locally $G \times_H V \times [0,\infty)^k$

$\circlearrowleft G$ trivially

<u>Claim</u>

$$A_G(X) \simeq \Sigma^\infty_G X_+ \times \underbrace{Wh_G(X)}_{}$$

G-spectrum w/

$\Omega^{\infty-1} Wh_G(M)^G \simeq H^\infty_G(M)$

when M cpt smth G-mfld

Thm (Browder - Quinn, Rothenberg)

M G-mfld $\dim M^H/W_H \geq 5$

$$\text{isom } \left. h\text{-cob} \middle/ \text{diffeo} \right. \cong Wh_G^{\mathrm{Diff}}(M)$$

$$\cong \bigoplus_{(H)} Wh\left(M_H / W_H \right)$$

Defn $\mathcal{H}_G(M)$: eqvt. h-cob space

$$\mathcal{H}_G^\infty(M) = \operatorname{colim}_V \mathcal{H}_G(M \times D(V)).$$

Thm (Goodwillie - Igusa - Malkiewich - M)

(in progress)

$\dim M_H / W_H \geq 5$

$$\mathcal{H}_G^{iso}(M) \simeq \prod_{(H)} \mathcal{H}(M_H / W_H)$$

stable result

$$\mathcal{H}_G^{\infty}(M) \simeq \prod_{(H)} \mathcal{H}^{\infty}(M^H_{h W_H})$$

Rmk $\quad (\Sigma_G^{\infty} X_+)^G$

$$\underset{\text{Tom-Dieck}}{\simeq} \bigvee_{(H)} \Sigma_+^{\infty} X^H_{\omega_H}$$

Q Can this space be described by eqvt alg K-theory?

Combined w/ previous work of Malkiewich - M

$$\Sigma^\infty_G M_+ \longrightarrow A_G(M) \longrightarrow Wh_G(M)$$

$$\Omega^\infty Wh_G(M)^G \simeq \mathcal{H}^\infty_G(M)$$

<u>Want</u> $A_G(X)^G \simeq \prod_{(H)} A(X^H_{hW_H})$

$\wr$ Rognes, Bodzitch-Dorabiola

X = G-space

$$K(\underbrace{R_G(X)})$$

retraction G-spaces/X

G-map

+ "genuine" Waldhausen struct.

in particular G-weak equiv

not $R(X)^{hG}$

Same objects & morphisms

but $R(X)^{hG}$ has "coarse"

Waldhausen structure.

Interlude on Spectral Mackey functors

Recall

Ab-enriched 1-obj cat $\longleftrightarrow$ ring R

Ab-enriched functor $\longleftrightarrow$ Module

$R \to Ab$ $\qquad$ M over R

Schwede-Shipley perspective

- Ab-enriched cat R $\longleftrightarrow$ ring on many objects

if $a,b,c \in \text{ob}\, R$

$$R_{a,b} \otimes R_{b,c} \to R_{a,c}$$

- Ab-enriched functor $\mapsto$ module M

$R \xrightarrow{M} Ab$ over R

$a \mapsto M_a$

& $R_{a,b} \otimes M_a \to M_b$

Recall A Mackey functor is a B_G-module

where B_G ring on object set $\{G/H\}$

& $B_G(G/H, G/K) \cong K_0$ (iso classes of spans
$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ G/H & & G/K \end{tikzcd}$$
)

Def A spectral Mackey functor is a module in Sp over B_G, ring in Sp on obj set $\{G/H\}$

w/ $B_G(G/H, G/K) = K\left(\text{cat of spans } \begin{tikzcd}[column sep=small] & S \arrow[dl] \arrow[dr] & \\ G/H & & G/K \end{tikzcd}\right)$

<u>Thm</u> BG-modules $\simeq$ G-Spectra

(Gillou–May,
Gillou–May–M–Osorno
Barwick, Glasman,
Shah, Nardin, Naumann,
Mathien, Noel)

Recall

$$\begin{tikzcd} K^P(\mathrm{Mfld}_n^{G,\partial}) \arrow[rr] \arrow[dr, "\text{captures } \chi_G \text{ on } \pi_0"'] & & K_G(\underline{\mathbb{Z}})^G \\ & A_G(*)^G \arrow[ur] & \end{tikzcd}$$

$\rightsquigarrow$ Coolle–Chen–Meji~

$\rightsquigarrow$ eqvt. linearization (- -)