



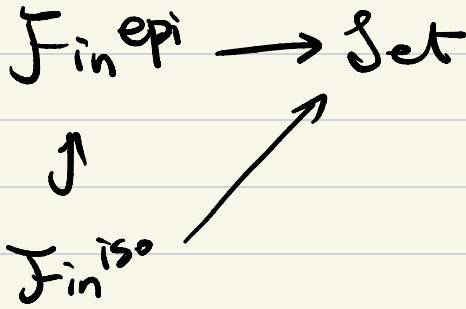
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G - Commutative monoids

Segal's approach:

$$\begin{array}{ccc} M \in \text{Set} & \Rightarrow & \text{Fin}^{\text{iso}} \rightarrow \text{Set} \\ & & \downarrow \tau \mapsto M^{\times T} \\ & & \text{Fin}^{\text{op}} \rightarrow \text{Set} \end{array}$$

$$\begin{array}{ccc} M \times M & \xrightarrow{m} & M \\ \text{"} & & \\ M^{\times \{a, b\}} & \rightarrow & M^{\times \{*\}} \\ & & \downarrow \begin{array}{l} | \\ \downarrow \text{ixm} \\ \downarrow \text{mx1} \\ M^{\times 2} \\ \downarrow \\ M \end{array} \\ M^{\times \{a, b\}} & \xrightarrow{\quad} & M \\ \tau \parallel \downarrow & \nearrow m & \\ M^{\times \{a, b\}} & & \end{array}$$



non-unital
commutative
monoid
structure
on M

$$M^{\times \emptyset} = * \longrightarrow M$$

$$\begin{array}{ccc}
 \emptyset \cup \{x\} & \xrightarrow{\cong} & \{x\} \\
 \downarrow & & \nearrow \\
 \{x, x\} & \longrightarrow & \{x\}
 \end{array}$$

$$\text{Fin}^{\text{iso}} \longrightarrow \text{Set}$$

$$\begin{array}{ccc}
 \text{Fin}^{\text{iso}} & \longrightarrow & \text{Set} \\
 \downarrow & & \nearrow \\
 \text{Fin} & & \text{Com. monoid structure on } M
 \end{array}$$

Combining those

can define

$A = \text{objects} - \text{finite sets}$

$$A(S, T) = \{S \leftarrow U \rightarrow T\} / \text{iso}$$

$\perp\!\!\!\perp$ is the categorical product

$$(S \perp\!\!\!\perp T \xrightarrow{S} S = S)$$

A functor $\underline{M}: A \rightarrow \text{Set}$ is product preserving

$$\textcircled{1} \underline{M}(\varphi) = *$$

$$\textcircled{2} \underline{M}(S \perp\!\!\!\perp T) \xrightarrow{\cong} \underline{M}(S) \times \underline{M}(T)$$

If M is product-preserving $\Rightarrow \forall T$

$$\begin{array}{ccc}
 M(T)^{\times 2} & \xleftarrow{\cong} & M(T \amalg T) \xrightarrow{M(\Delta)} M(T) \\
 & \searrow \cong & \nearrow \\
 & & M(T)
 \end{array}$$

\Rightarrow (Product Preserving functors)

$\downarrow \cong$
 Comm. Monoid

$\begin{array}{c} M \\ \downarrow \\ M(*) \end{array}$

Elementar

$$\text{Top } G \longrightarrow \text{Fun}(G^{\text{op}}, \text{Top})$$

Coefficient system of spaces

$$\leadsto \text{Fun}(G^{\text{op}}, \text{Set})$$

$$\widehat{\text{Fun}}(G^{\text{op}}, \text{Comm}) = \text{Coeff systems}$$

$$\pi_0 : \text{Fun}(G^{\text{op}}, \text{Top}) \rightarrow \widehat{\text{Fun}}(G^{\text{op}}, \text{Set})$$

Have an inclusion

$$G_H \hookrightarrow G_K$$

$$H/K \hookrightarrow G/K$$

Not full

\Rightarrow get a restriction map

$$\iota_H^* : \text{Coef}^G \rightarrow \text{Coef}^H$$

Has a left adjoint Ind_H^G

a right adjoint co-Ind_H^G

$\text{Fin}^G =$ finite cprod completion of \mathcal{C}_G

$$\Rightarrow \text{Coef}^G \simeq \text{Fun}^\Pi(\text{Fin}_G^{\text{op}}, \mathbb{T})$$

$$\iota_H^*(\underline{M})(T) = \underline{M}(G \times_H T)$$

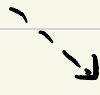
$$\text{co-Ind}_H^G(\underline{M})(T) = \underline{M}(\iota_H^* T)$$

Example

$$\underline{M}(C_p/C_p)$$

$$\downarrow \text{res}_e^{C_p}$$

$$\underline{M}(C_p/\xi_{e^2})$$



$$M(C_p/\xi_{e^2})^{C_p}$$



C_p

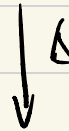
\otimes



$$C_p \times M$$

$$\text{Ind}_e^{C_p} M$$

M



$$M^{C_p} = \text{Fun}(C_p, M)$$

$$\text{CoInd}_e^{C_p} M$$

$$(\text{CoInd}_H^G \text{C}_H^* \underline{M})(T) \cong \underline{M}(G/H \times T)$$

get for any Coef. system M

$$\text{Fin}^{G, \text{iso}} \rightarrow \text{Coef}$$

$$S \mapsto \underline{M}(S \times -)$$

$$\begin{aligned} S_0 \sqcup S_1 &\mapsto \underline{M}((S_0 \sqcup S_1) \times -) \\ &\cong \underline{M}(S_0 \times -) \times \underline{M}(S_1 \times -) \end{aligned}$$

An extension over Fin^G ?

Example $G = C_p$.

$* \amalg * \rightarrow *$ $\left\{ \begin{array}{l} \underline{M}(T) \text{ is a non-unital com.} \\ \text{monoid } \forall T \end{array} \right.$

$\phi \rightarrow *$
unit

f
 \downarrow
 $\underline{M}(f) = \underline{M}(T) \rightarrow \underline{M}(S)$

$S \xrightarrow{f} T \Rightarrow$

is a map in Com.

$C_p \rightarrow *$

$\underline{M}(C_p \times -) \rightarrow \underline{M}(-)$

$\underline{M}(C_p/e) \xrightarrow{T} \underline{M}(C_p/C_p)$

$\Delta \downarrow$

$\downarrow \text{res}$

$\text{Map}(C_p, \underline{M}(C_p/e)) \xrightarrow{+} \underline{M}(C_p/e)$

$$f \longrightarrow C_p \rightarrow \sum_{g \in G} g f(g)$$

$$\text{Res}_e^{C_p} \circ T(m) = \sum_{g \in C_p} g \cdot m$$

If we ask for compatibility w/ restriction

to all subgroups \Rightarrow

an extension to $\text{Fin } G$ is to give a

Mackey functor restricted to \underline{M}

Segal: in Top w/ \simeq , get a model of

\mathbb{E}_∞ -spaces

Shimakawa Top^G , get a model of

G - \mathbb{E}_∞ spaces

In G -comm. monoids, always get \times is

cat coprod, catInd_H^G is left adjoint

to V_H^* ,

Get a \otimes ing over (finite) G -sets

$$\textcircled{1} (S \amalg T) \otimes \underline{M} = (S \otimes \underline{M}) \oplus (T \otimes \underline{M})$$

$$\textcircled{2} \quad (G/H) \otimes \underline{M} = \text{Coh}_{G/H} \mathcal{U}_H^* \underline{M}$$

\Rightarrow get an extension of $G\text{-Conn}(\underline{M}, \underline{N}) \in \text{Set}$
to a coefficient system of Set.

$$\begin{aligned} G\text{Conn}(\underline{M}, \underline{N})(G/H) &= H\text{Conn}(\mathcal{U}_H^* \underline{M}, \mathcal{U}_H^* \underline{N}) \\ &\simeq G\text{Conn}(G/H \otimes \underline{M}, \underline{N}) \end{aligned}$$

\Rightarrow get tensoring of simplicial $G\text{Conn}$
monoids over simplicial G -sets

\Rightarrow Bredon homology additively

Multiplicatively \Rightarrow Loday type construction

An indexing cat for G is a wide,
pullback stable, finite coproduct complete

Subcat of $\text{Fin } G$ $\text{Fin } G$

$$\underline{\text{Ex.}} \quad \text{Fin } G, \text{ Fin } G^* = \left| \begin{array}{l} f: S \rightarrow T \text{ s.t.} \\ f \text{ preserves isotopy:} \\ \text{Stab}(f(s)) \stackrel{?}{=} \text{Stab}(s) \\ \text{(isovariant)} \end{array} \right.$$

Stability in an unstable world

Last Time: G -commutative monoids

$$G\text{-Comm}(\text{Coef}^G) \cong \text{Mackey}^G$$

Terms described

$$N_H^G : \text{Mackey}^H \rightarrow \text{Mackey}^G$$

$$\Pi : \text{Mackey}^G \times \text{Mackey}^G \rightarrow \text{Mackey}^G$$

Definition..

$$\begin{array}{ccc}
 \begin{array}{l} \text{spans} \\ \text{of} \\ \text{fin } G \end{array} \swarrow & A^G \times A^G & \xrightarrow{\underline{M \times N}} \text{Set} \\
 & \downarrow \times & \nearrow \\
 & A^G & \xrightarrow{\underline{M \sqcup N}} \text{Set}
 \end{array}$$

Lan

$$\begin{array}{ccc}
 A^H & \xrightarrow{\underline{M}} & \text{Set} \\
 \text{Map}^H(G, -) \downarrow & & \nearrow \\
 A^G & \xrightarrow{\underline{N_H^G M}} & \text{Set}
 \end{array}$$

$$\underline{N_H^G M} \cong \underline{\Pi_0(N_H^G H M)}$$

A com. monoid for \square

• Green functor:

① $\forall G/H, \quad \underline{R}(G/H)$ for com. ring

② $f: G/H \rightarrow G/K$

$\Rightarrow f^*: \underline{R}(G/K) \rightarrow \underline{R}(G/H)$ com. rings

$f_*: \underline{R}(G/H) \rightarrow \underline{R}(G/K)$

map of $\underline{R}(G/K)$ -mod

* double coset

Frobenius Relation

$$a \cdot \text{tr}_H^K(b) = \text{tr}_H^K(\text{res}_H^K(a) \cdot b)$$

M \square N

- ① forget coef systems
- ② form levelwise tensors
- ③ form free Mackey generated
- ④ Impose Frobenius relation

Then $G\text{-Comm}(\text{Mackey } G) \cong \text{Jamb } G$

Aside $\text{Jamb } G$.

Green functor $R +$

Maps of Green functors

$$N_H^K \text{ } \underset{H}{\text{CH}}^* R \rightarrow \underset{K}{\text{C}}^* R$$

$$N_H^K \underline{R} \rightarrow \underline{R}$$

C_p . Get the following structure

$$\begin{array}{c} \underline{R}(C_p/C_p) \\ \text{res} \downarrow \quad \uparrow \text{tr}_e^{C_p} \\ \underline{R}(C_p/e) \\ \hookrightarrow C_p \end{array}$$

$N_{C_p}^{C_e}$

this w/

1) res + tr ∈ Mackey

2) res + n is G-Con (Quot)
w/ X

$$N_e^{G_p}(a+b) = N_e^G(a) + N_e^{G_p}(b) + \text{tr}(\dots)$$

Tambara's TNR functors

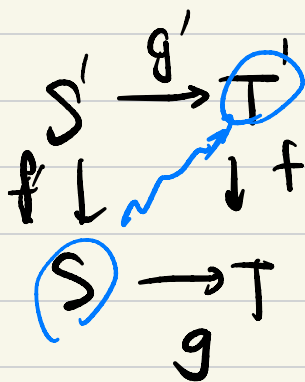
$$P(S, T) = \left\{ S \xleftarrow{f} U \xrightarrow{g} U \xrightarrow{h} T \right\}$$

↓
finite G-sets

→ sets of multi-sets → sets of monomials → add them up

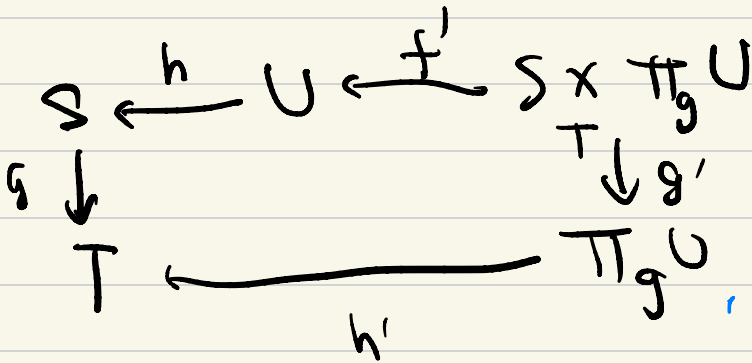
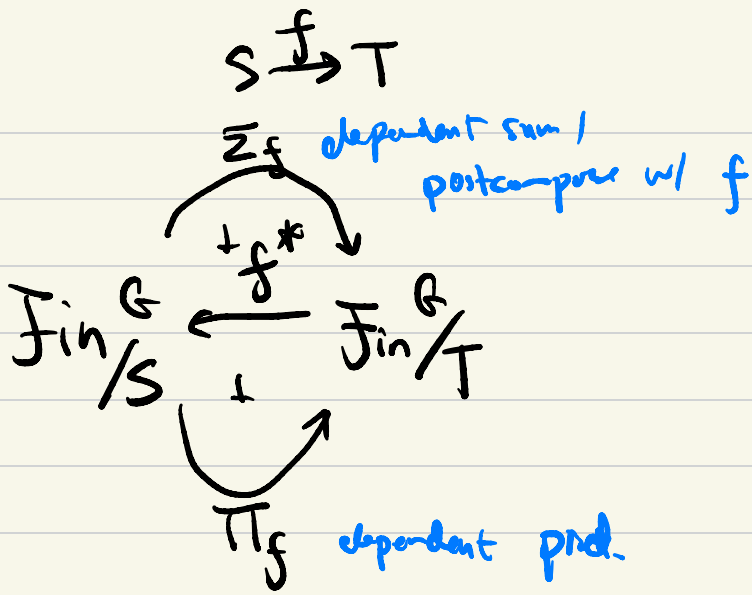
$T_h \cdot N_g \cdot R_f$

If we have a pb diagram



$$R_f \circ T_g = T_{g'} \circ R_{f'} \quad \text{res + tr}$$

$$R_f \circ N_g = N_{g'} \circ R_{f'} \quad \text{res + n}$$



$$N_{g'} \circ T_h = T_{h'} \circ N_{g'} \circ R_{f'}$$

take $T = x$

evaluate

i)

$$\begin{array}{ccccccc} G/H & \xleftarrow{\Delta} & G/H & \sqcup & G/H & \xleftarrow{\quad} & G/H \times \text{Map}^H(G, \{a, b\}) \\ \pi \downarrow & & & & & & \downarrow \\ * & & & & & & \text{Map}^H(G, \{a, b\}) \end{array}$$

$$N_{\pi} \cdot T_{\Delta}(a, b) = N_H^G(a+b)$$

ii)

$$\begin{array}{ccccccc} G/H & \xleftarrow{\varrho} & G/K & \xleftarrow{\quad} & G/H \times \text{Map}^H(G, H/K) \\ \downarrow & & & & \downarrow \\ * & & & & \text{Map}^H(G, H/K) \end{array}$$

$$N_H^G \text{tr}_K^H(a) = \gamma$$

$$\text{Fin}^G \otimes \text{Jamb}^G$$

$$G/H \otimes R := \left(N_H^G \right) i_H^* R$$

left adjoint to

$$i_H^*: \text{Jamb}^G \rightarrow \text{Jamb}^H$$

$$\Rightarrow \text{Jamb}^G(\underline{R}, \underline{S}) = \underline{\text{Jamb}}(\underline{R}, \underline{S})(G/G)$$

$$\text{Jamb}^H(i_H^* R, i_H^* S) = \underline{\text{Jamb}}(\underline{R}, \underline{S})(G/H)$$

//

$$\text{Jamb}^G(N^{G/H} R, \underline{S})$$

If \underline{R} is a Tambara functor, then

an \underline{R} -module is a module in Mackey^G - the underlying Green functor

A module is a Strictland, Cont sys of
= abelian groups
on

$$\text{Ab} \left(\text{Tamb}^G_{\underline{R}} = \text{Tambara argument to } \underline{R} \right) \text{Tamb}^G_{\underline{R}}$$

get an embedding $\text{Full Tamb}^{\text{op}}_{\underline{R}} \cdot \text{Ab}$

$$\text{Tamb}^G_{\underline{R}} \xrightarrow{h(-)} \text{P} \left(\text{Tamb}^G_{\underline{R}} \right)$$

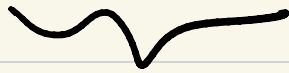
$$\underline{S} \mapsto \left(\underline{B} \mapsto \text{Tamb}^G_{\underline{R}}(\underline{B}, \underline{S}) \right)$$

By Yoneda Lemma

functorial lift of h_S to $\mathcal{A}b$

$$\iff \mathcal{A}b \times \mathcal{S} \xrightarrow{+} \mathcal{S}$$

$$\mathcal{R} \xrightarrow{+0} \mathcal{S}$$



$$\mathcal{S} = \mathcal{R} \oplus \mathcal{I}$$

$\mathcal{Q} \mathcal{I}$ is a nonunital

Tambura functor

Abelian grp:

$$\underline{\mathbb{I}} \square \underline{\mathbb{I}} \xrightarrow{0} \underline{\mathbb{I}}$$

\mathbb{R} -Mod = GCorn (Strickland's)

= Mackey (Jamb/\mathbb{R})

$$C_p : \underline{\mathbb{I}}(C_p/C_p) \xrightarrow{n} \underline{\mathbb{I}}(C_p/e) \xrightarrow{\text{tr}} C_p$$

$\downarrow \text{res}_e^C$

+ ① $\text{res}_e^C(n(a)) = 0$
② n is an additive map

③ n is a map
 from $R(C_p/\mathbb{Z})\text{-Mod}$
 $\rightarrow R(C_p/K_p)\text{-Mod}$

R G - \mathbb{E}_∞ -obj in S_p^G

$G\text{-Comm}/R \rightsquigarrow S_p^G(G\text{-Comm}/R)$
 \Downarrow
 $S_p(G\text{-Comm}/R)$ $\begin{matrix} 12 \\ R\text{-Mod} \end{matrix}$
 stable ∞ -cat