Michael Hill

G-Commutative monoids

Segal's approach:

$$
\begin{tikzcd}
M \in \mathrm{Set} \arrow[r, Rightarrow] & \mathrm{Fin}^{iso} \arrow[r] \arrow[d, dashed, bend right] & \mathrm{Set} \\
& T \longmapsto M^{\times T} & \\
& \mathrm{Fin}^{op} \arrow[r] & \mathrm{Set}
\end{tikzcd}
$$

$$
\begin{tikzcd}
M \times M \arrow[r, "m"] \arrow[d, equal] & M \\
M^{\times\{a,b\}} \arrow[r] & M^{\times\{*\}}
\end{tikzcd}
\qquad
\begin{tikzcd}
M^{\times 3} \arrow[d, shift right, "1\times m"'] \arrow[d, shift left, "m\times 1"] \\
M^{\times 2} \arrow[d] \\
M
\end{tikzcd}
$$

$$
\begin{tikzcd}
M^{\times\{a,b\}} \arrow[r] \arrow[d, "\tau"', "\cong"] & M \\
M^{\times\{a,b\}} \arrow[ur, "m"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Fin}^{epi} \arrow[r] & \mathrm{Set} \\
\mathrm{Fin}^{iso} \arrow[u, hook] \arrow[ur] &
\end{tikzcd}
$$

non-unital commutative monoid structure on M

$$
M^{\times \emptyset} = * \longrightarrow M
$$

$$
\begin{tikzcd}
\emptyset \sqcup \{*\} \arrow[d, hook] \arrow[dr, "\cong"] & \\
\{*, *\} \arrow[r] & \{*\}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Fin}^{iso} \arrow[r] \arrow[d, hook] & \mathrm{Set} \\
\mathrm{Fin} \arrow[ur] &
\end{tikzcd}
$$

com. monoid structure on M

Combining those

can define

$\mathcal{A}$ = objects · finite sets

$$\mathcal{A}(S,T) = \{ S \leftarrow U \rightarrow T \} /_{\text{iso}}$$

$\amalg$ is the categorical product

$$\left(\begin{tikzcd} & S \arrow[dl] \arrow[dr, equal] & \\ S \amalg T & & S \end{tikzcd} \right)$$

A functor $\underline{M} : \mathcal{A} \to \mathrm{Set}$ is product preserving

- $\underline{M}(\varnothing) = *$
- $\underline{M}(S \amalg T) \xrightarrow{\cong} \underline{M}(S) \times \underline{M}(T)$

If $\underline{M}$ is product-preserving $\Rightarrow \forall T$

$$\begin{tikzcd} \underline{M}(T)^{\times 2} \arrow[rr, bend right, "m"'] & \underline{M}(T \amalg T) \arrow[l, "\cong"'] \arrow[r, "\underline{M}(\nabla)"] & \underline{M}(T) \end{tikzcd}$$

$\Rightarrow$ (Product Preserving functors) $\xrightarrow{\cong}$ Comm. Monoid

$$\underline{M} \mapsto \underline{M}(*)$$

Elmendorff

$$\mathrm{Top}^G \longrightarrow \mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top})$$

Coefficient system of spaces

$\rightsquigarrow \mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Set})$

$\mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Comm}) =$ Coeff systems

$$\pi_0 : \mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Top}) \to \mathrm{Fun}(\mathcal{O}_G^{op}, \mathrm{Set})$$

Have an inclusion

$$\mathcal{O}_H \hookrightarrow \mathcal{O}_G$$
$$H/K \longmapsto G/K$$

<u>Not</u> full

$\Rightarrow$ get a restriction map

$$\iota_H^* : \mathrm{Coef}^G \to \mathrm{Coef}^H$$

Has a left adjoint Ind_H^G

a right adjoint coInd_H^G

Fin^G = finite coprod completion of $\mathcal{O}_G$

$$\Rightarrow \quad \mathrm{Coef}^G \simeq \mathrm{Fun}^{\pi}(\mathrm{Fin}_G^{op}, -)$$

$$\iota_H^*(\underline{M})(T) = \underline{M}(G \times_H T)$$

$$\mathrm{CoInd}_H^G(\underline{M})(T) = \underline{M}(\iota_H^* T)$$

Example

$$\begin{tikzcd}
\underline{M}(C_p/C_p) \arrow[d, "\mathrm{res}^{C_p}_{e}"] \arrow[dr, dashed] & \\
\underline{M}(C_p/\{e\}) \arrow[r, phantom, "\cong"] & M(C_p/\{e\})^{C_p}
\end{tikzcd}$$

$\underline{M}(C_p/\{e\}) \circlearrowleft C_p$

$$\begin{tikzcd}
\phi \arrow[d] & M \arrow[d, "\Delta"] \\
C_p \times M & M^{\times C_p} = \mathrm{Fun}(C_p, M)
\end{tikzcd}$$

$\mathrm{Ind}_e^{C_p} M \qquad \mathrm{CoInd}_e^{C_p} M$

$$(\mathrm{CoInd}_H^G \, i_H^* \underline{M})(T) \cong \underline{M}(G/H \times T)$$

get for any coef. system $\underline{M}$

$$\mathcal{F}in^{G,\mathrm{iso}} \longrightarrow \mathrm{Coef}$$

$$S \longmapsto \underline{M}(S \times -)$$

$$S_0 \amalg S_1 \longmapsto \underline{M}((S_0 \amalg S_1) \times -)$$

$$\cong \underline{M}(S_0 \times -) \times \underline{M}(S_1 \times -)$$

An extension over $\mathcal{F}in^G$?

Example $G = C_p$.

$* \amalg * \longrightarrow *$ $\Big\}$ $\underline{M}(T)$ is a non-unital com. monoid $\forall T$

$\emptyset \to *$
unit

$S \xrightarrow{f} T \Rightarrow$

$\underline{M}(f) : \underline{M}(T) \longrightarrow \underline{M}(S)$

is a map in Com.

$C_p \longrightarrow *$

$\underline{M}(C_p \times -) \longrightarrow \underline{M}(-)$

$$\begin{tikzcd}
\underline{M}(C_p/e) \arrow[r, "T"] \arrow[d, "\Delta"'] & \underline{M}(C_p/C_p) \arrow[d, "\mathrm{res}"] \\
\mathrm{Map}(C_p, M(C_p/e)) \arrow[r, "+"'] & M(C_p/e)
\end{tikzcd}$$

$$\begin{array}{c}\circlearrowright\\ C_p\\ |\end{array}$$

$$f \longrightarrow C_p \to \sum_{g\in G} g f(g)$$

$$\mathrm{Res}^{C_p}_{e} \circ T(m) = \sum_{g \in C_p} g\cdot m$$

If we ask for compatibility w/ restriction to all subgroups $\Rightarrow$ an extension to Fin^G is to give a Mackey functor restricted to $\underline{M}$

Segal: in Top w/ $\simeq$, get a model of

$\mathbb{E}_\infty$-spaces

Shimakawa $\mathcal{T}op^G$, get a model of

G-$\mathbb{E}_\infty$ spaces

In G-comm. monoids, always get $\times$ is

cat coprod, CoInd_H^G is left adjoint

to ι_H^*.

Get a $\otimes$ing over (finite) G-sets

① $(S \amalg T) \otimes \underline{M} = (S \otimes \underline{M}) \oplus (T \otimes \underline{M})$

② $(G/H) \otimes \underline{M} = \mathrm{Coind}_H^G i_H^* \underline{M}$

$\Rightarrow$ get an extension of $G\text{-}\mathrm{Comm}(\underline{M}, \underline{N}) \in \mathrm{Set}$

to a coefficient system of Set.

$$G\mathrm{Comm}(\underline{M}, \underline{N})(G/H) = H\mathrm{Comm}(i_H^* \underline{M}, i_H^* \underline{N})$$
$$\simeq G\mathrm{Comm}(G/H \otimes \underline{M}, \underline{N})$$

$\Rightarrow$ get tensoring of simplicial GCom monoids over simplicial G-sets

$\Rightarrow$ Bredon homology additively

Multiplicatively $\Rightarrow$ Loday type construction

An indexing cat for G is a wide, pullback stable, finite coprod complete subcat of $\mathcal{F}in^G$ $\quad \mathcal{F}in^G_{\mathcal{O}}$

Ex. $\mathcal{F}in^G$, $\mathcal{F}in^G_{\mathcal{O}^*} = \left\{ \begin{array}{l} f: S \to T \text{ s.t.} \\ f \text{ preserves isotropy:} \end{array} \right.$

$\mathrm{Stab}(f(s)) \overset{?}{=} \mathrm{Stab}(s)$

(isovariant)

Stability in an unstable World

Last Time: G - commutative monoids

$$G\text{-Comm}(\mathrm{Corf}^G) \simeq \mathrm{Mackey}^G$$

Teena described

$$N_H^G : \mathrm{Mackey}^H \longrightarrow \mathrm{Mackey}^G$$

$$\boxtimes : \mathrm{Mackey}^G \times \mathrm{Mackey}^G \longrightarrow \mathrm{Mackey}^G$$

Definition.

spans of finG

$$\begin{tikzcd} \mathcal{A}^G \times \mathcal{A}^G \arrow[r, "\underline{M} \times \underline{N}"] \arrow[d, "\times"'] & \mathrm{Set} \\ \mathcal{A}^G \arrow[ur, dashed, "\underline{M} \boxtimes \underline{N}"'] & \end{tikzcd}$$

Lan

$$\begin{tikzcd} \mathcal{A}^H \arrow[r, "\underline{M}"] \arrow[d, "\mathrm{Map}^H(G,-)"'] & \mathrm{Set} \\ \mathcal{A}^G \arrow[ur, dashed, "N_H^G \underline{M}"'] & \end{tikzcd}$$

$$N_H^G \underline{M} \simeq \pi_0 \left(N_H^G H\underline{M} \right)$$

$\underline{A}$ com. monoid for $\square$

- Green functor :

① $\forall\, G/H$, $\underline{R}(G/H)$ com. ring

② $f : G/H \to G/K$

$\Rightarrow f^* : \underline{R}(G/K) \to \underline{R}(G/H)$ com. rings

$f_* : \underline{R}(G/H) \to \underline{R}(G/K)$

map of $\underline{R}(G/K)$-mod

+ double coset

Frobenius Relation

$$a \cdot tr_H^K(b) = tr_H^K(res_H^K(a) \cdot b)$$

$\underline{M} \square \underline{N}$

① forget coef systems

② form levelwise tensors

③ form free Mackey generated

④ Impose Frobenius relation

<u>Thm</u> $G\text{-}Comm(Mackey^G) \simeq Jamb^G$

<u>Aside</u> $Jamb^G$.

Green functor $\underline{R}$ +

Maps of Green functors

$$N_H^K i_H^* R \longrightarrow i_K^* R$$

$$N^{K/H} \underline{R} \longrightarrow \underline{R}$$

C_p : Get the following structure

$$\begin{tikzcd}
\underline{R}(C_p/C_p) \arrow[d, "\mathrm{res}"'] \\
\underline{R}(C_p/e) \arrow[u, bend left=60, "n_e^{C_p}"] \arrow[u, bend right=60, "\mathrm{tr}_e^{C_p}"'] \arrow[loop below, "C_p"]
\end{tikzcd}$$

this w/

1) res + tr ∈ Mackey

2) res + n is G-Comm (Set)

w/ $\times$

$$n_e^{C_p}(a+b) = n_e^{C_p}(a) + n_e^{C_p}(b) + tr(\cdot \ \cdot)$$

Tambara's TNR functors

$$P(S,T) = \{ S \xleftarrow{f} U \xrightarrow{g} V \xrightarrow{h} T \}$$

finite G-sets

⇢ sets of multi sets ⇢ sets of monomials ⇢ add them up

$$T_h \cdot N_g \cdot R_f$$

If we have a p b diagram

$$
\begin{tikzcd}
S' \arrow[r, "g'"] \arrow[d, "f'"'] & T' \arrow[d, "f"] \\
S \arrow[r, "g"'] \arrow[ur, squiggly] & T
\end{tikzcd}
$$

$$R_f \circ T_g = T_{g'} \circ R_{f'}$$ res + tr

$$R_f \circ N_g = N_{g'} \circ R_{f'}$$ res + n

$$S \xrightarrow{f} T$$

$$\begin{tikzcd}
\mathcal{F}in^G_{/S} \arrow[rr, bend left=60, "\Sigma_f"] \arrow[rr, phantom, bend left=30, "\perp"] \arrow[rr, bend right=60, "\Pi_f"'] \arrow[rr, phantom, bend right=30, "\perp"] & & \mathcal{F}in^G_{/T} \arrow[ll, "f^*"']
\end{tikzcd}$$

Σ_f dependent sum / postcompose w/ f

Π_f dependent prod.

$$\begin{tikzcd}
S \arrow[d, "g"'] & U \arrow[l, "h"'] & S \times \Pi_g U \arrow[l, "f'"'] \arrow[d, "g'"] \\
T & & \Pi_g U \arrow[ll, "h'"]
\end{tikzcd}$$

$$N_g \circ T_h = T_{h'} \circ N_{g'} \circ R_{f'}$$

take $T = *$

i)

$$
\begin{tikzcd}
G/H \arrow[d, "\pi"'] & G/H \sqcup G/H \arrow[l, "\nabla"'] & & G/H \times \mathrm{Map}^H(G, \{a,b\}) \arrow[ll, "\text{evaluate}"'] \arrow[d] \\
* & & & \mathrm{Map}^H(G, \{a,b\}) \arrow[lll]
\end{tikzcd}
$$

$$N_\pi \cdot T_\nabla(a,b) = N_H^G(a+b)$$

ii)

$$
\begin{tikzcd}
G/H \arrow[d] & G/K \arrow[l, "\varepsilon"'] & & G/H \times \mathrm{Map}^H(G, H/K) \arrow[ll] \arrow[d] \\
* & & & \mathrm{Map}^H(G, H/K) \arrow[lll]
\end{tikzcd}
$$

$$N_H^G \, tr_K^H(a) = \, ?$$

$\mathrm{Fin}^G \circledast \mathrm{Tamb}^G$ $\quad G/H \otimes \underline{R} := N_H^G i_H^* \underline{R}$

left adjoin to

$i_H^* : \mathrm{Tamb}^G \to \mathrm{Tamb}^H$

$$\Rightarrow \mathrm{Tamb}^G(\underline{R}, \underline{S}) = \underline{\mathrm{Tamb}}(\underline{R}, \underline{S})(G/G)$$

$$\mathrm{Tamb}^H(i_H^* \underline{R}, i_H^* \underline{S}) = \underline{\mathrm{Tamb}}(\underline{R}, \underline{S})(G/H)$$

$$\|$$

$$\mathrm{Tamb}^G(N^{G/H} \underline{R}, \underline{S})$$

If $\underline{R}$ is a Tambara functor, then an $\underline{R}$-module is a module in Mackey^G the underlying Green functor

A module à la Strickland,

$$\mathrm{Ab}\left(\mathrm{Tamb}^G_{/\underline{R}} = \begin{array}{c}\text{Tambara} \\ \text{augmented to } \underline{R}\end{array}\right) \stackrel{\text{Cat sys of abelian groups on } \mathrm{Tamb}^G_{/\underline{R}}}{=} \mathrm{Fun}(\mathrm{Tamb}^{op}_{/\underline{R}}, \mathrm{Ab})$$

get an embedding

$$\mathrm{Tamb}^G_{/\underline{R}} \xrightarrow[h_{(-)}]{} P(\mathrm{Tamb}^G_{/\underline{R}})$$

$$\underline{S} \longmapsto (\underline{B} \longmapsto \mathrm{Tamb}_{/\underline{R}}(\underline{B}, \underline{S}))$$

By Yoneda Lemma

functorial lift of $h_{\underline{S}}$ to Ab

$$\iff \quad \underline{S} \times_{\underline{R}} \underline{S} \xrightarrow{+} \underline{S}$$

$$\underbrace{\underline{R} \xrightarrow{\underline{0}} \underline{S}}$$

$$\underline{S} = \underline{R} \oplus \underline{I}$$

& $\underline{I}$ is a nonunital

Tambara functor

Abelian grp:

$$\underline{I} \boxtimes \underline{I} \xrightarrow{\phi} \underline{I}$$

$\underline{R}$-Mod = GComm (Strickland's)

= Mackey(Tamb/$\underline{R}$)

C_p:
$$\begin{tikzcd}
\underline{I}(C_p/C_p) \arrow[d, shift right, "\mathrm{res}^{C_p}_e"'] \\
\underline{I}(C_p/e) \arrow[u, shift right, "\mathrm{tr}"'] \arrow[u, bend left=80, "n"] \arrow[loop below, "C_p"]
\end{tikzcd}$$

+ ① $\mathrm{res}^{C_p}_e(n(a)) = 0$

② n is an additive map

? η is a map from $R(C_p/e)\text{-Mod} \to R(C_p/C_p)\text{-Mod}$

R G-E_∞-alg in Sp^G

$$\begin{tikzcd}
G\text{-Comm}/R \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & Sp^G(G\text{-Comm}/R) \arrow[d, phantom, "\cong" rotate=90] \\
Sp(G\text{-Comm}/R) & R\text{-Mod}
\end{tikzcd}$$

stable ∞-cat