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Topological modular forms and bilinear forms

TMF E_∞-ring spectrum

$$\pi_{2t}\mathrm{TMF} \xrightarrow{\phi} MF_t$$

integral modular form of wt t

- ϕ is rationally an iso
- ϕ is not injective nor surjective
  ($\eta \in \pi_1 \mathrm{TMF}, \cdots$) ($\Delta \in MF_t$)

Q Are theta functions in $\mathrm{im}(\phi)$?

A (Hopkins, Bockstalt) Yes

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1 Modular forms

A modular form of wt k

is a holomorphic function $\mathbb{H} \xrightarrow{f} \mathbb{C}$

s.t. $f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ + ...

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rightsquigarrow f(\tau+1) = f(\tau)$

$\rightsquigarrow f(\tau) = \sum_{n \geq -N} a_n q^n$ q-expansion

$(q = e^{2\pi i \tau})$

Integral modular form - mod form s.t. $a_n \in \mathbb{Z}$.

MF_d - integral m.f. of wt d

Examples

1) Let $Q : \mathbb{Z}^d \to \mathbb{Z}$ be a quadratic form.

and let $b(r, w) = B_Q$ be the assoc bilinear form.

$$\vartheta_Q(\tau) = \sum_{v \in \mathbb{Z}^d} q^{Q(v)} = \sum_{n \geq 0} \#\{v \in \mathbb{Z}^d \mid Q(v) = n\} q^n$$

$n \in \mathbb{Z}$?

modular form of wt $d/2$ if b unimodular and positive definite

2) $\Delta \in MF_{12}$ ($\Delta^{-1} \in MF_{-12}$)

Geometric reformulation

If $k=0 \rightsquigarrow$ mod forms of wt 0

are functions on $\mathbb{H}/SL_2(\mathbb{Z})$

$\mathbb{C}/\langle 1,\tau\rangle \leftrightarrow \tau \quad \cong$ iso. classes of ell curves

In general, mod forms of w.t. are

sections of a line bundle on

$[\mathbb{H}/SL_2\mathbb{Z}]$, $\omega^{\otimes k}$

$\shortparallel$

$\mathcal{M}_{ell,\mathbb{C}}$ (complexified) modular stack of elliptic curves

<u>Thm</u> (q-expansion principle)

$$MF_k = H^0(\mathcal{M}_{ell}, \omega^{\otimes k})$$

global sections

modular stack of all curves

Topological modular forms

Goerss – Hopkins – Miller:

$\exists$ sheaf $\mathcal{O}^{top}$ of E_∞-ring spectra

on $\mathcal{M}_{ell}$

TMF := global sections of $\mathcal{O}^{top}$

$$\pi_t \mathcal{O}^{top} = \begin{cases} 0 & \text{if } t \text{ odd} \\ \omega^{\otimes t/2} & \text{if } t \text{ even} \end{cases}$$

Descent SS

$$E_2^{s,t} = H^s(\mathcal{M}_{ell}, \pi_t \mathcal{O}^{top})$$

$$\Rightarrow \pi_{t-s} TMF$$

Edge homomorphism

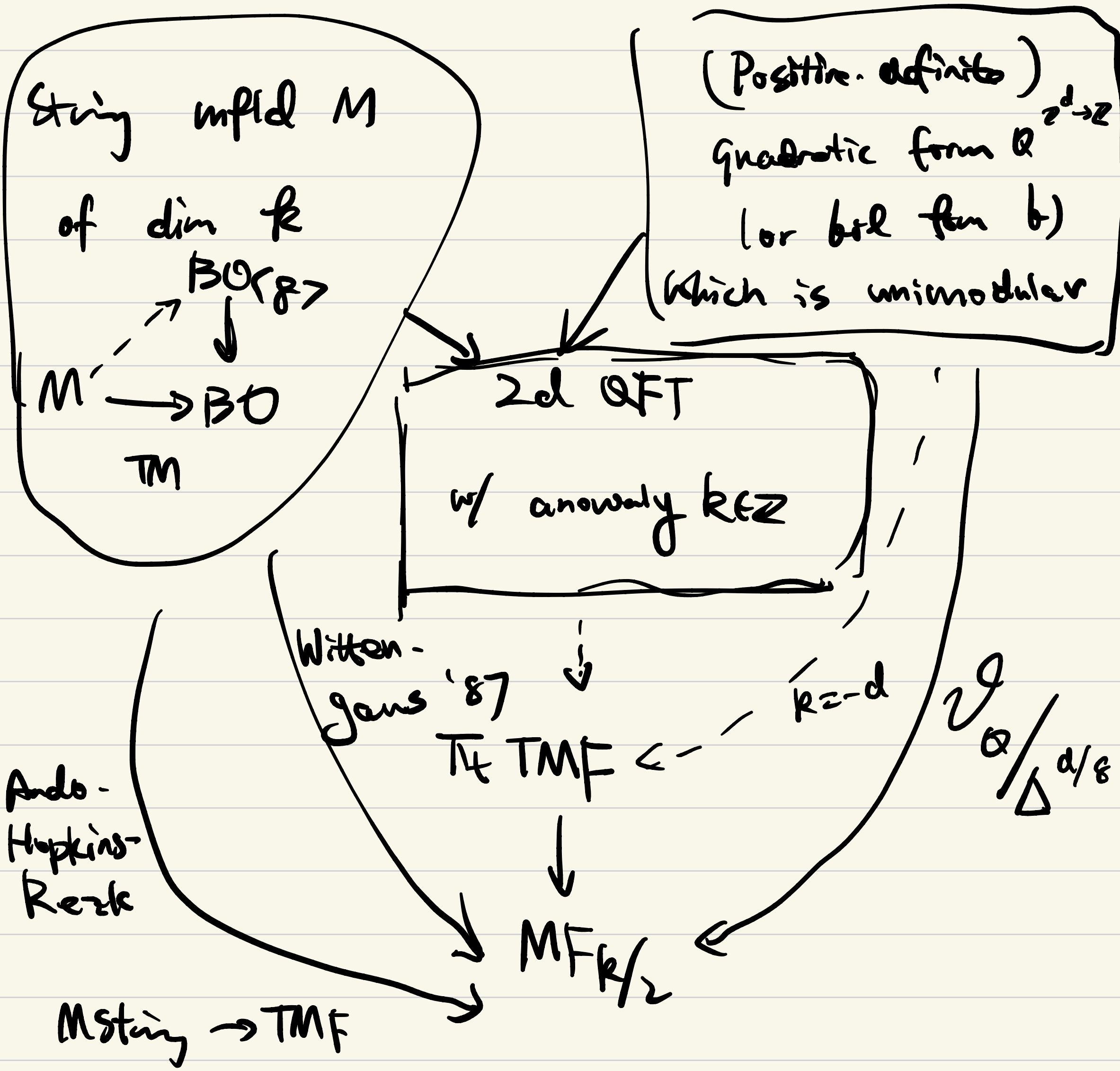
$$\pi_t TMF \longrightarrow H^0(\mathcal{M}_{ell}, \pi_t \mathcal{O}^{top})$$

$$= \begin{cases} 0 & \text{if } t \text{ odd} \\ H^0(\mathcal{M}_{ell}, \omega^{\otimes t/2}) & \text{if } t \text{ even} \end{cases}$$

$$H^0(\mathcal{M}_{ell}, \omega^{\otimes t/2}) = MF_{t/2}$$

3) TMF and QFT

(Witten, Segal, Stolz, Teichner)



Goal Define for every sym$^{\text{unimodular}}$ bilinear form $b : \mathbb{Z}^d \otimes \mathbb{Z}^d \to \mathbb{Z}$ a class in $\pi_* TMF$ s.t. the image in MF_* is ϑ_Q if b comes from pos. def. Q.

$$\text{Quadr Forms} \hookrightarrow \text{Bil. Forms}$$
$$Q \longmapsto B_Q$$

$$\tfrac{1}{2} b(v,v) \dashleftarrow b$$

Eichler–Zagier

4. Jacobi forms and multilinear theta forms

<u>Def</u> (Jacobi form) Let b be a sym. bil form $\mathbb{Z}^d \otimes \mathbb{Z}^d \to \mathbb{Z}$

A <u>Jacobi form</u> of wt k and index b is a holom. function

$$\mathbb{H} \times \mathbb{C}^d \longrightarrow \mathbb{C}$$

satisfying some transf behavior dependent on k, b, ...

wrt $SL_2\mathbb{Z} \times \mathbb{Z}^{2d}$-action on

$\mathbb{H} \times \mathbb{C}^d$.

$\underline{Ex}$ If Q pos def, unimod.

$$\Theta_Q(\tau, z) = \sum_{v \in \mathbb{Z}^d} q^{Q(v)} e^{2\pi i\, b(v,z)}$$

$$\rightsquigarrow \vartheta_Q = \Theta_Q(-, 0)$$

$JF^{\mathbb{C}}_{*,b}$ is free over $MF^{\mathbb{C}}_*$ of rank 1

on Θ_Q

<u>Plan</u> Define for every b, topological

Jacobi form TJF

Show that for b unimodular,

$$
\begin{tikzcd}
\mathrm{TJF}_b \simeq \mathrm{TMF}[?] \arrow[d] & \text{generator} \arrow[dl, mapsto] \\
\mathrm{TMF} & \text{element in } \pi_* \mathrm{TMF}
\end{tikzcd}
$$