Hermitian objects..

$(\mathcal{C}, Ϙ)$ Hermitian ∞-cat

$He(\mathcal{C}, Ϙ)$

$(x \in \mathcal{C}, \; q \overset{\infty}{\in} \Omega^{\infty} Ϙ(x))$

$Fn(\mathcal{C}, Ϙ) := He(\mathcal{C}, Ϙ)^{\simeq}$

Poincaré objects

$(\mathcal{C}, Ϙ)$ Poincaré ∞-cat

(meaning we have $D: \mathcal{C} \xrightarrow{\sim} \mathcal{C}^{op}$
w/ $\hom_{\mathcal{C}}(X, DY) \simeq B_Ϙ(X, Y)$)

(x, q) Poincaré object

$x \in \mathcal{C}, \; q \overset{\infty}{\in} \Omega^{\infty} Ϙ(x)$ s.t. $q_{\#}: x \xrightarrow{\sim} Dx$

$$\begin{gathered} \downarrow \\ B_Ϙ(X, X) \\ \simeq \\ \hom_{\mathcal{C}}(X, DX) \end{gathered}$$

$\mathrm{Pn}(\mathcal{C}, \text{Ϙ})$

$\mathrm{hyp}(X)$ for $X \in \mathcal{C}$ is Poincaré

w/ $\mathrm{hyp}(X) = X \oplus DX$

$$\text{Ϙ}_{hyp}: \begin{tikzcd} \text{Ϙ} \arrow[r] \arrow[dr] & \text{Ϙ}(X \oplus DX) \\ & B_{\text{Ϙ}}(X, DX) \arrow[u] \end{tikzcd}$$

$$B_{\text{Ϙ}}(X, DX) \simeq \hom_{\mathcal{C}}(X, X)$$

$$\mathrm{Pn}(\mathrm{Hyp}(\mathcal{C})) \simeq \mathcal{C}^{\simeq}$$

$$\mathrm{Hyp}(\mathcal{C}) = (\mathcal{C} \oplus \mathcal{C}^{op}, \text{Ϙ}_{hyp})$$

$$\text{Ϙ}_{hyp}(X, Y) = \hom_{\mathcal{C}}(X, Y)$$

D_{hyp} = switch X, Y.

Cobordism $(X,q) \to (X',q')$ in $\mathcal{P}_n(\mathcal{C},\mathfrak{P})$

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X & & X' \end{tikzcd}$$

w/ $\eta : \alpha^* q \xrightarrow{\sim} \beta^* q'$

s.t. $W \xrightarrow{\sim} DX \times_{DW} DX'$.

<u>cobordant</u>.

<u>metabolic</u> : cobordant to 0.

$L \to X$, null homotopy of $q|_L$

$\quad\ \ \vdots$

$\quad\ \ q$ s.t.

$$L \to X \xrightarrow{\sim} DX \to DL$$

is exact.

Q - construction

$(\mathcal{C}, Ϙ)$ Hermitian

$\mathcal{Q}_n(\mathcal{C}) \subset \mathrm{Fun}(\mathrm{TwAr}[n], \mathcal{C})$

spanned by those φ,

$$\begin{tikzcd}
\varphi_{il} \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp", very near start] \arrow[dr, phantom, "\ulcorner", very near end] & \varphi_{jl} \arrow[d] \\
\varphi_{ik} \arrow[r] & \varphi_{jk}
\end{tikzcd}$$

$(i \leq j \leq k \leq l)$

$$Ϙ_n(\varphi) = \lim_{\mathrm{TwAr}[n]^{op}} Ϙ\varphi^{op}$$

$\mathcal{Q}_n(\mathcal{C}, Ϙ) \in \mathrm{Cat}_\infty^h$,

$(\mathcal{C}, Ϙ) \in \mathrm{Cat}_\infty^p$

$\Rightarrow \mathcal{Q}_n(\mathcal{C}, Ϙ) \in \mathrm{Cat}_\infty^p$

$$\mathcal{Q}_0(\mathcal{C}, \mathfrak{Q}) = (\mathcal{C}, \mathfrak{Q})$$

$$\mathcal{Q}_1(\mathcal{C}, \mathfrak{Q}) = \{\text{spans}\}$$

$$\mathrm{He}\,\mathcal{Q}_1(\mathcal{C}, \mathfrak{Q}) = \left\{ \begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X & & Y \end{tikzcd}, q, q' \text{ w/ } h: \alpha^* q \simeq \beta^* q' \right\}.$$

$$P_n\,\mathcal{Q}_1(\mathcal{C}, \mathfrak{Q}) = \{\text{cobordisms between } (X, q), (X', q') \in P_n(\mathcal{C}, \mathfrak{Q})\}$$

$$P_n\,\mathcal{Q}_n(\mathcal{C}, \mathfrak{Q}) = \{\text{a sequence of } n \text{ composable cobordisms}\}$$

$(\mathcal{C}, \mathfrak{Q})$ Poincaré

$P_n\mathcal{Q}_\bullet(\mathcal{C}, \mathfrak{Q})$ is complete

Segal space.

$\mathrm{Cob}(\mathcal{C}, Ϙ)$ = ∞-cat corresponding to $\mathrm{Pn}\,\mathcal{Q}_\bullet(\mathcal{C}, Ϙ^{[1]})$

$\mathrm{Cob}(\mathrm{Hyp}(\mathcal{C})) \simeq \mathrm{Span}(\mathcal{C})$.

L-group

$$L_n(\mathcal{C}, Ϙ) := \pi_0 |\mathrm{Cob}(\mathcal{C}, Ϙ^{[-n-1]})|$$

(of cobordism classes of Poincaré objects in $(\mathcal{C}, Ϙ^{[-n]})$)

$L_n(\mathcal{D}^p(R), Ϙ^q_M) \cong L^q_n(R, M)$

(Wall-Ranicki quadratic L-groups)

$L_n(\mathrm{Hyp}(\mathcal{C})) \simeq 0.$

$\mathcal{GW}$ space

$\mathcal{GW}(\mathcal{C}, Ϙ)$

$:= \Omega |\mathrm{Cob}(\mathcal{C}, Ϙ)|$

$= \Omega |\mathcal{P}_n\, Q_\cdot(\mathcal{C}, Ϙ^{[1]})|$

$\mathcal{GW}(\mathrm{Hyp}(\mathcal{C})) \simeq \Omega |\mathrm{Span}(\mathcal{C})| = \mathcal{K}(\mathcal{C}).$

$$GW^r(R, M) \simeq GW^r_{cl}(R, M)$$

(using unimodular forms & group completion)

for $r \in \{gq, gs, ge\}$

but not q & s.

$$GW_0(\mathcal{C}, \Qoppa) = \{[X, q] \in \mathrm{Pn}(\mathcal{C}, \Qoppa)\} / \sim$$

$[X, q] \sim [\mathrm{hyp}(L)]$

(X, q) hyperbolic w/ Lagrangian $L \to X$

$$K_0(\mathcal{C})_{C_2} \to GW_0(\mathcal{C}, \Qoppa) \to L_0(\mathcal{C}, \Qoppa) \to 0$$

Cat_∞^p is complete & cocomplete

$\mathrm{Cat}_\infty^p \xrightarrow{\text{forget}} \mathrm{Cat}_\infty^{ex}$ preserves small limits / colimits.

Cat_∞^p is semi-additive

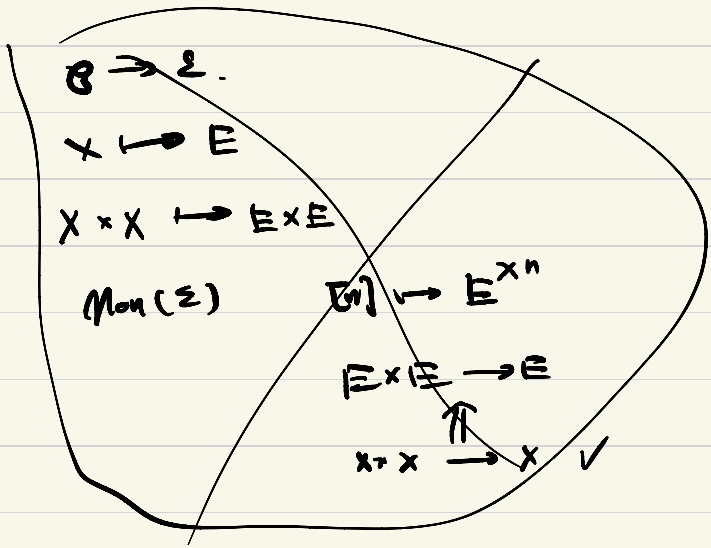
(has zero object, $\amalg = \Pi$)

$$\begin{tikzcd}
 & \mathrm{Mon}_{E_\infty}(\mathcal{E}) \arrow[d] \\
\mathrm{Cat}_\infty^p \arrow[ur, dashed] \arrow[r, "\text{prod. preserving}"] & \mathcal{E}
\end{tikzcd}$$

$\mathcal{F} : \mathrm{Cat}^{p}_{\infty} \longrightarrow \mathcal{E}$

is group like

if $\mathrm{Cat}^{p}_{\infty} \longrightarrow \mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{E})$ is so



$(\mathcal{C}, Ϙ) \mapsto \mathcal{GW}(\mathcal{C}, Ϙ) = \Omega|\mathrm{Cob}(\mathcal{C}, Ϙ)|$

is group-like.

while $P_n(-)$ is not.

(split) Poincaré-Verdier sequence

if

$$(\mathcal{C}, \Omega) \longrightarrow (\mathcal{D}, \Phi) \to (\mathcal{E}, \Psi)$$

vanishes

and is fiber cofiber sequence

in Cat^p_∞.

(split) P-V injection/projection

Metabolic Poincaré ∞-Cat

$(\mathcal{C}, \Omega) \in \mathrm{Cat}^p_\infty$

$\mathrm{Met}(\mathcal{C}, \Omega) \in \mathrm{Cat}^p_\infty$

consists of $L \to X$ in $\mathcal{C}$

$$\Omega_{met}([L \to X]) := \mathrm{fib}(\Omega(X) \to \Omega(L))$$

$$D_{met} = \mathrm{fib}[DX \to DL] \to DX$$

$$\mathrm{He}(\mathrm{Met}(\mathcal{C}, \Omega)) \leftrightsquigarrow (X, q) \in \mathrm{He}(\mathcal{C}, \Omega), \quad L \to X, \; q|_L \overset{\eta}{\sim} 0$$

it's Poincaré iff $(X, q) \in \mathrm{Pn}(\mathcal{C}, \Omega)$.

η Lagrangian L.

split P-V seq

$$(\mathcal{C}, \mathfrak{P}^{[-1]}) \to \mathrm{Met}(\mathcal{C}, \mathfrak{P}) \to (\mathcal{C}, \mathfrak{P})$$

$$[L \to X] \longmapsto X$$

$X = 0 \rightsquigarrow L \in \mathcal{C}$

$\mathfrak{P}([L \to X]) = \mathrm{fib}(\mathfrak{P}(X) \to \mathfrak{P}(L))$

$X = 0, \quad = \Omega \mathfrak{P}(L)$

$\mathrm{fib}(\mathfrak{P}X \to \mathfrak{P}L) \to \mathfrak{P}(X)$ ✓

$L \mapsto [L \to 0]$? ✓

$\mathrm{Map}(X, Y) \simeq \mathrm{Map}([L \to X], ?)$

$fD \qquad Df$

$L \to X \rightsquigarrow \mathrm{fib}(DX \to DL) \to DX$

$\rightsquigarrow DX.$ ✓

$L \to X \rightsquigarrow X$

$\rightsquigarrow DX$

$L \rightsquigarrow \Omega DL$

$\rightsquigarrow \Omega DL \to 0$

$L \rightsquigarrow L \to 0$

$\rightsquigarrow \mathrm{fib}(0 \to DL) \to 0$

$D^{[-1]}$

$B_{\mathfrak{P}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, DY).$

$B_{\mathfrak{P}^{[-1]}}(X, Y) \simeq \Sigma^{-1} B_{\mathfrak{P}}(X, Y)$

$\simeq B_{\mathfrak{P}}(X, \Sigma Y) \qquad B(\Sigma X, Y)$

$\simeq \mathrm{Hom}_{\mathcal{C}}(X, D\Sigma Y) \qquad \mathrm{Hom}_{\mathcal{C}}(\Sigma X, DY)$

$\mathrm{Hom}_{\mathcal{C}}(X, \Omega DY).$

$D^{[-1]} = D\Sigma^{-1}.$

$\Omega B = 0 \prod_{B(X,Y)} 0$

$$\begin{tikzcd}
\Omega B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
0 \arrow[r] & B
\end{tikzcd}$$

$0 = B(X, 0) \prod_{B(X,Y)} B(X, 0)$

$= B(X, 0 \amalg_Y 0)$

$B = B$

$B(X, \cdot) : \mathcal{C}^{op} \to \mathrm{Sp}$

(split) P–V square :

Cartesian square w/ (split) P–V v-legs.

Verdier-localizing:

sends P-V sq to fiber sq

additive:

sends P-V split sq to fiber sq.

$\mathcal{P}_n : \mathrm{Cat}^{p}_{\infty} \to \mathcal{S}$ is V-localizing

$$\begin{tikzcd}
& (\mathcal{C},\Omega) \arrow[d, "s"'] \arrow[dr, "\mathrm{id}"] & \\
\mathrm{Met}(\mathcal{C},\Omega) \arrow[r] \arrow[dr] & Q_1(\mathcal{C},\Omega) \arrow[r, "d_1"] \arrow[d] & (\mathcal{C},\Omega) \\
& \mathrm{Hpy}(\mathcal{C}) &
\end{tikzcd}$$

split P–V seq

What if $\mathcal{F} : \mathrm{Cat}_\infty^p \to \mathcal{E}$ is group-like additive functor?

$$(\mathcal{C}, \mathcal{Q}) \longmapsto \mathcal{GW}(\mathcal{C}, \mathcal{Q})$$

is additive

fibration thm.

$\mathrm{Cob} : \mathrm{Cat}_\infty^p \to \mathcal{S}$

sends split P-V projection to bicart fib.

Barwick :

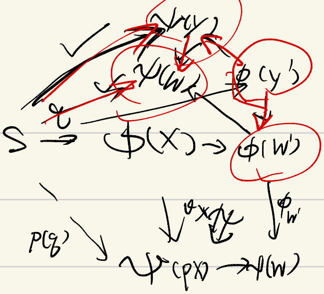
$p : \mathcal{D} \to \mathcal{E}$ exact

admits fully faithful left and right adjoints

then

$p_* : \mathrm{Span}(\mathcal{D}) \longrightarrow \mathrm{Span}(\mathcal{E})$ is bicart fib.

$$\begin{tikzcd}
(\mathcal{D}, \Phi) \arrow[r, "p"] & (\mathcal{E}, \psi) \\
(Y', r') \arrow[r, dashed] & (Y, r) \\
(W', \eta') \arrow[u] \arrow[r, dashed] \arrow[d, "\alpha'"'] & (W, \eta) \arrow[u] \arrow[d] \\
(X, q) \arrow[r, dashed] & (p(X), p(q))
\end{tikzcd}$$



$\Phi \Rightarrow p^*\psi$

$\alpha' : W' \to X$.

Cor of Fibration Thm

$GW(-)$ additive

$$\Rightarrow \quad GW(\mathrm{Met}(\mathcal{C}, \Xi)) \simeq GW(\mathrm{Hyp}(\mathcal{C})) \simeq K(\mathcal{C}).$$

$$GW(Q, (\mathcal{C}, \Xi)) \simeq GW(\mathcal{C}, \Xi) \times K(\mathcal{C})$$

$$GW(\mathcal{C}, \Xi^{[-1]}) \xrightarrow{\text{fgt}} K(\mathcal{C}) \xrightarrow{\text{hyp}} GW(\mathcal{C}, \Xi)$$

(space level Bott–Genauer seq)

$(\mathcal{C}, \Omega^{[-1]}) \longrightarrow \mathrm{Met}(\mathcal{C}, \Omega) \longrightarrow (\mathcal{C}, \Omega).$

what if spectral level?

$$\mathcal{U}(\mathcal{C}, \Omega) := \mathrm{fib}[\, K(\mathcal{C}) \xrightarrow{hyp} GW(\mathcal{C}, \Omega)]$$

$$\mathcal{V}(\mathcal{C}, \Omega) := \mathrm{fib}[\, GW(\mathcal{C}, \Omega) \xrightarrow{fgt} K(\mathcal{C})].$$

Karoubi's fundamental thm for GW.

(on)

$$\mathcal{V}(\mathcal{C}, \Omega) \simeq \Omega\, \mathcal{U}(\mathcal{C}, \Omega^{[2]}).$$

$$\simeq \Omega\, GW(\mathcal{C}, \Omega^{[1]})$$

$$(D^p(R), (\Omega^q_M)^{[2]}) \simeq (D^p(R), \Omega^q_{-M})$$

$$\Rightarrow \mathcal{V}^q(R, M) \simeq \Omega\, \mathcal{U}^q(R, -M).$$

similarly holds for Ω^s_M,

$$(D^p(R), (\Omega^{gq}_M)^{[2]}) \simeq (D^p(R), \Omega^{ge}_{-M})$$

Similarly holds for

$\mathcal{I}^{\mathcal{g}_e}_{M}$ $\mathcal{I}^{g^s}_{\sim M}$

$(\mathcal{C}, \mathfrak{Q})$ Poincaré.

$\mathcal{A} \subset \mathcal{C}$ is called isotropy if:

- $\mathfrak{Q}|_{\mathcal{A}}$ vanishes
- one has "cofree" functor $\mathcal{C} \to \mathcal{A}$.

$\mathcal{A} \subset \mathcal{A}^{\perp} = \{ y \in \mathcal{C} \mid B_{\mathfrak{Q}}(x, y) = 0 \ \forall x \in \mathcal{A} \}$

<u>Verdier quotient</u>

$$\mathrm{Hlgy}(\mathcal{A}) := \mathcal{A}^{\perp}/\mathcal{A} \quad (\in \mathrm{Cat}_{\infty}^{p}).$$

called the homology of $\mathcal{A}$.

$\mathcal{A}$ is called a Lagrangian if

$$\mathrm{Hlgy}(\mathcal{A}) = 0.$$

$\mathcal{Q}n(\mathcal{C}, \mathfrak{Q})$ has an isotropy subcat consisting

$- \{ \ 0 \leftarrow 0 \rightarrow 0 \leftarrow 0 \rightarrow 0 \cdots 0 \leftarrow W \xrightarrow{\sim} X \quad w/$

homology $\mathcal{Q}_{n-1}(\mathcal{C}, \Omega)$.

$\mathcal{A} \subset \mathcal{C}$ isotropy.

$\mathcal{F} : \mathrm{Cat}_\infty^p \to \mathcal{E}$ grouplike additive.

Then

$$\mathcal{F}(\mathcal{C}, \Omega) \simeq \mathcal{F}(\mathrm{Hlgy}(\mathcal{A})) \times \mathcal{F}(\mathrm{Hyp}(\mathcal{A}))$$

$$\mathcal{F}(\mathcal{Q}_n(\mathcal{C}, \Omega)) \simeq \mathcal{F}(\mathcal{C}, \Omega) \times \mathcal{F}(\mathrm{Hyp}(\mathcal{C}))^n.$$

$$\mathcal{GW}(\mathcal{Q}_n) \simeq \mathcal{GW}(\mathcal{C}, \Omega) \times \mathcal{K}(\mathcal{C})^n.$$

$$\mathcal{C} \xrightarrow{i} \mathcal{D} \underset{r}{\overset{p}{\rightleftarrows}} \mathcal{E}$$

fiber seq in Cat_∞^{ex}

$p \dashv r$, r fully faithful

Then $\mathrm{Hyp}(\mathcal{D})$ has Lagrangian

$$(i, r^{op}) : \mathcal{C} \times \mathcal{E}^{op} \longrightarrow \mathcal{D} \times \mathcal{D}^{op}.$$

Cor

$$\mathcal{F}(\mathrm{Hyp}(\mathcal{D})) \simeq \mathcal{F}(\mathrm{Hyp}(\mathcal{C} \times \Sigma^{op}))$$

$$\simeq \mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \times \mathcal{F}(\mathrm{Hyp}(\Sigma)).$$

Cor (Waldhausen additivity)

$$K(\mathcal{D}) \simeq K(\mathcal{C}) \times K(\Sigma).$$

Cobordisms cats w/ coefficients

$\mathcal{F} : \mathrm{Cat}_\infty^{p} \to \mathcal{S}$ additive.

$(\mathcal{C}, Ϙ)$ Poincaré

then $\mathcal{F}\,\mathcal{Q}_\bullet(\mathcal{C}, Ϙ^{[1]})$ is Segal

$\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, Ϙ)$

and complete if $\mathcal{F}$ is limit preserving.

$\mathrm{Cob} = \mathrm{Cob}^{\mathcal{P}_n}$.

$\mathcal{F}$ group-like :

$\mathcal{F}\mathcal{Q}_\bullet(\mathcal{C}, Ϙ^{[1]}) \simeq \mathcal{F}(\mathcal{C}, Ϙ^{[1]}) \times \mathcal{F}(\mathrm{Hyp}(\mathcal{C}))^\bullet$

is groupoid object

$\Updownarrow$

$\mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \curvearrowright \mathcal{F}(\mathcal{C}, Ϙ^{[1]})$

induced by Poincaré Functor

$\mathrm{Hyp}(\mathcal{C}) \to (\mathcal{C}, Ϙ^{[1]}), \ (X, Y) \mapsto X \oplus D_Ϙ Y.$

then $\mathcal{F}\mathcal{Q}_\circ(\mathcal{C}, \mathfrak{P})$ is complete $\Leftrightarrow$ $\mathcal{F}(\mathrm{Ltyp}(\mathcal{C})) = 0$

Generalized Fibration Thm.

$\mathcal{F} : \mathrm{Cat}^{p}_{\infty} \to \mathcal{S}$ additive.

$\mathrm{Cob}^{\mathcal{F}}(-)$ sends split P-V projections to bicart fib.

<u>Cor</u> $\mathcal{F}$ additive $\Rightarrow$ $|\mathrm{Cob}^{\mathcal{F}}(-)|$ additive.

$$\begin{tikzcd}
X \arrow[r, mapsto] & {[0 \leftarrow X \to 0]} & \\
(\mathcal{C}, \mathfrak{P}) \arrow[r] \arrow[d] & \mathcal{Q}_1(\mathcal{C}, \mathfrak{P}^{[1]}) \arrow[d] & ? \\
0 \arrow[r] & (\mathcal{C}, \mathfrak{P}^{[1]}) \times (\mathcal{C}, \mathfrak{P}^{[1]}) &
\end{tikzcd}$$

induces $\mathcal{F}(\mathcal{C}, \mathfrak{P}) \to \Omega |\mathrm{Cob}^{\mathcal{F}}(\mathcal{C}, \mathfrak{P})|$

if $\mathcal{F}$ group-like $\Rightarrow$ $|\mathrm{Cob}^{\mathcal{F}}(-)|$ is so,

and above $\mathcal{F} \xRightarrow{\sim} \Omega |\mathrm{Cob}^{\mathcal{F}}(-)|$.

$\mathcal{F} : \mathrm{Cat}^{p}_{\infty} \to \mathcal{S}$ additive, group-like

$\mathrm{Cob}^{\mathcal{F}} : \mathrm{Cat}^{p}_{\infty} \to \mathrm{Sp}$

$(\mathcal{C}, \mathfrak{Q}) \mapsto (\mathcal{F}_0(\mathcal{C}, \mathfrak{Q}), \mathcal{F}_1(\mathcal{C}, \mathfrak{Q}), \dots)$.

$\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_n = |\mathrm{Cob}^{\mathcal{F}_{n-1}}(-)|$.

$$\mathcal{F}_n(\mathcal{C}, \mathfrak{Q}) \xrightarrow{\sim} \Omega \mathcal{F}_{n+1}(\mathcal{C}, \mathfrak{Q}).$$

$\mathrm{Cob}^{\mathcal{F}}$ is additive $\Omega^{\infty} \mathrm{Cob}^{\mathcal{F}} \simeq \mathcal{F}$

(delooping)

GW spectrum functor

$\mathrm{GW} : \mathrm{Cat}^{p}_{\infty} \to \mathrm{Sp}$

$(\mathcal{C}, \mathfrak{Q}) \mapsto \mathrm{Cob}^{\mathrm{GW}}(\mathcal{C}, \mathfrak{Q})$.

(Hermitian analogue of alg-K spectrum $K(\mathcal{C})$)

$\mathrm{GW}(\mathcal{C}, \mathfrak{Q})$ isn't connective in general

? it's equivalent to Schlichting's defn in some localizing variant the setting of exact cats

$$GW(\mathrm{Hyp}(\mathcal{C})) \simeq K(\mathcal{C}).$$

Bott-Genauer seq (sp ver.)

$$GW(\mathcal{C}, \mathfrak{Q}^{[-1]}) \xrightarrow{fgt} K(\mathcal{C}) \xrightarrow{hyp} GW(\mathcal{C}, \mathfrak{Q}).$$

Cor for $i<0$, $\pi_i K(\mathcal{C}) = 0$

so $GW_i(\mathcal{C}, \mathfrak{Q}) \cong GW_{i-1}(\mathcal{C}, \mathfrak{Q}^{[-1]})$.

By induction, $n<0$

$$GW_n(\mathcal{C}, \mathfrak{Q}) \cong \mathrm{coker}[K_0(\mathcal{C}) \to GW_0(\mathcal{C}, \mathfrak{Q}^{[n]})]$$

$$\cong L_0(\mathcal{C}, \mathfrak{Q}^{[n]})$$

$$= L_n(\mathcal{C}, \mathfrak{Q}).$$

<u>Summary</u> : $GW_n = L_n$ for $n<0$.

Universality for $\mathcal{GW}$.

$P_n \Rightarrow \mathcal{GW}$ exhibits $\mathcal{GW}$ as the initial group-like additive functor to $\mathcal{S}$ under P_n.

for GW.

$\Sigma^\infty P_n \Rightarrow GW$.

$\mathcal{F} \longmapsto |\mathrm{Cob}^{\mathcal{F}}(-)|$ realizes suspension in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{p}}_{\infty}, \mathcal{S})$

($\mathcal{Q}$ is Σ)

dual $\mathcal{Q}$ - construction $\mathcal{F} \mapsto \mathcal{F}\mathcal{Q}^n$

(right adjoint to $\mathcal{F} \longmapsto \mathcal{F}\mathcal{Q}_n$)

L-Theory space

$(\mathcal{C}, \Omega) \in \mathrm{Cat}_\infty^p$,

$$\mathcal{P}_n(\mathcal{C}, \Omega) = (\mathrm{Fun}(\mathcal{J}_n^{op}, \mathcal{C}), \Omega_{[n]})$$

$\mathcal{J}_n$ = Poset of nonempty $S \subset [n]$.

$\Omega_{[n]} : \varphi \longmapsto \lim_{\mathcal{J}_n} \Omega \circ \varphi$.

$\mathcal{P}_\bullet(\mathcal{C}, \Omega) : \Delta \to \mathrm{Cat}_\infty^p$.

suppose $\mathcal{F} : \mathrm{Cat}_\infty^p \to \mathcal{E}$, $\mathcal{E}$ admits geometric

realizations.

$$\rho\mathcal{F} : \mathrm{Cat}_\infty^p \longrightarrow \mathcal{E},$$

$$(\mathcal{C}, \Omega) \longmapsto |\mathcal{F}\mathcal{P}_\bullet(\mathcal{C}, \Omega)|.$$

(Lurie - Ranicki)

$$\mathcal{L} := \rho\, \mathrm{Pn}.$$

it's group-like Verdier-localizing.

$\pi_n \mathcal{L} \cong L_n$ for $n \geq 0$.

$SW := Q P_n$

$$\begin{tikzcd} P_n \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathcal{L} \\ SW \arrow[ur, Rightarrow, "\nexists"'] & \end{tikzcd}$$

$SW \rightsquigarrow \eta_n : J_n^{op} \to \mathrm{TwAr}[n]$

$T \longmapsto \min(T) \leq \max(T)$.

$\Downarrow$

(Poincaré)
Hermitian functor

$Q_n(\mathcal{C}, Ϙ) \to \rho_n(\mathcal{C}, Ϙ)$.

$\Downarrow$

$|\mathrm{Cob}(\mathcal{C}, Ϙ)| \simeq |P_n Q_\bullet(\mathcal{C}, Ϙ^{[1]})|$

$\downarrow$

$|P_n \rho_\bullet(\mathcal{C}, Ϙ^{[1]})| \simeq \mathcal{L}(\mathcal{C}, Ϙ^{[1]})$

$\Downarrow$

$SW(\mathcal{C}, Ϙ) \to \Omega \mathcal{L}(\mathcal{C}, Ϙ^{[1]}) \overset{\partial}{\simeq} \mathcal{L}(\mathcal{C}, Ϙ)$

$$(\mathcal{C}, Ϙ) \to \mathrm{Met}(\mathcal{C}, Ϙ) \to (\mathcal{C}, Ϙ^{[1]})$$

$$\Downarrow \mathcal{L}$$

$$\boxed{\Omega\mathcal{L}(\mathcal{C}, Ϙ^{[1]})} \dashrightarrow \mathcal{L}(\mathcal{C}, Ϙ) \to \mathcal{L}(\mathrm{Met}(\mathcal{C}, Ϙ)) \to \mathcal{L}(\mathcal{C}, Ϙ^{[1]})$$

$\mathcal{L}$ is <u>bordism invariant</u>

Internal Functor Cat

$(\mathcal{C}, Ϙ), (\mathcal{D}, \Phi) \in \mathrm{Cat}^h_\infty$.

$$\mathrm{Fun}^{ex}((\mathcal{C}, Ϙ), (\mathcal{D}, \Phi)) := (\mathrm{Fun}^{ex}(\mathcal{C}, \mathcal{D}), \mathrm{nat}^{\Phi}_{Ϙ})$$

$$\mathrm{nat}^{\Phi}_{Ϙ}(f) = \mathrm{nat}(Ϙ \Rightarrow f^*\Phi).$$

it's Poincaré if $(\mathcal{C}, Ϙ), (\mathcal{D}, \Phi)$ are so.

and its Poincaré objects are Poincaré functors.

$(f, \eta), (g, \vartheta) : (\mathcal{C}, Ϙ) \to (\mathcal{D}, \Phi)$ are called

cobordant if $\exists$ cobordism between them

$$(\text{in } \mathcal{P}n\,\mathrm{Fun}^{ex}((\mathcal{C}, Ϙ), (\mathcal{D}, \Phi))).$$

(f, η) is called a <u>bordism equivalence</u> if

exists "inverse" up to cobordism.

$(\mathcal{C}, Ϙ) \in \mathrm{Cat}^p_\infty$, $A \subset \mathcal{C}$ isotropy.

$\mathrm{Hlgy}(A) \to (\mathcal{C}, Ϙ)$ is b.e..

bordism invariant : sends b.e. to equivs.

Criterion for group-like additive $\mathcal{F}$ being b.i.

TFAE:

- $\mathcal{F}$ b.i..
- $\mathcal{F}$ vanishes on metabolic Poincaré ∞-cat.
- $\mathcal{F}$ vanishes on $\mathrm{Hyp}(\mathcal{C})$, $\forall \mathcal{C} \in \mathrm{Cat}^{ex}_\infty$.

Cor $\mathcal{L}$ is b.i..

(as $\mathrm{Pn}(\rho.\mathrm{Hyp}(\mathcal{C})) \simeq \mathrm{Fun}(\mathcal{T}^{op}, \mathcal{C})^{\simeq}$

simplicial subdivision of $\mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq}$.

$\mathcal{L}(\mathrm{Hyp}(\mathcal{C})) \simeq |\mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq}| \simeq |\mathcal{C}| \simeq *$.

L-spectrum of $(\mathcal{C}, Ϙ)$.

$$L(\mathcal{C}, Ϙ) = \tau_{\geq 0}\, b^{\mathcal{L}}(\mathcal{C}, Ϙ).$$

L: $\mathrm{Cat}^{p}_{\infty} \to \mathrm{Sp}$ is additive.

$\mathcal{L}$ - preserves $\mathrm{Hyp}(\mathcal{C}) \Rightarrow L$ b.i..

$\mathcal{GW} \to \mathcal{L}$ induce $GW \to L$.

$$L(\mathcal{C}, Ϙ^{[-1]}) \to L(\mathrm{Met}(\mathcal{C}, Ϙ)) \to L(\mathcal{C}, Ϙ).$$
$$\simeq 0$$

$$\Rightarrow L(\mathcal{C}, Ϙ^{[-1]}) \simeq \Omega L(\mathcal{C}, Ϙ).$$

$L(\mathcal{C}, Ϙ) \longleftrightarrow \Omega$-spectrum

$$\mathcal{L}(\mathcal{C}, Ϙ), \mathcal{L}(\mathcal{C}, Ϙ^{[1]}), \mathcal{L}(\mathcal{C}, Ϙ^{[2]}), \dots$$

w/ $\mathcal{L}(\mathcal{C}, Ϙ^{[n]}) \xrightarrow{\simeq} \Omega \mathcal{L}(\mathcal{C}, Ϙ^{[n+1]})$.

bondification :

$b : \mathcal{F} \to \mathcal{F}'$ between additive

$\mathrm{Cat}_\infty^p \to \mathrm{Sp}$.

b exhibits $\mathcal{F}'$ as the bondification

of $\mathcal{F}$

if $\mathcal{F}'$ is b.i.,

$\forall$ other b.i. additive $\mathcal{G} : \mathrm{Cat}_\infty^p \to \mathrm{Sp}$.

$\mathrm{nat}(\mathcal{F}', \mathcal{G}) \xrightarrow{\sim} \mathrm{nat}(\mathcal{F}, \mathcal{G})$.

$$GW^{bord} : Cat^{p}_{\infty} \to Sp$$

additive

$$GW^{bord} := \mathrm{cofib}(K_{hC_2} \to GW).$$

additive

induced from $hyp : K \to GW$.

GW^{bord} is additive since Sp is stable.

$GW \to GW^{bord}$ exhibits GW^{bord} as the bordification of GW.

<u>Indeed</u>

$$GW^{bord} \simeq L$$

<u>Cor.</u>

$$K_{hC_2} \to GW \to L.$$

Proof

$$
\begin{tikzcd}
 & K_{hC_2} \arrow[r] & GW \arrow[r] \arrow[d, Rightarrow, "\rho"] & GW^{bord} \\
 & K_{hC_2} \arrow[r] \arrow[d] & GW \arrow[r] \arrow[d] & GW^{bord} \arrow[d, "\simeq"] \\
(\rho K)_{hC_2} \arrow[r, "\simeq"] & \rho(K_{hC_2}) \arrow[r] & \rho GW \arrow[r] & \rho GW^{bord}
\end{tikzcd}
$$

$(\rho K)_{hC_2} \simeq 0$, $\quad \rho GW \simeq L$

Cor $GW \Rightarrow L$ exhibits L as initial bordism invariant functor under GW.

Cor $\Sigma^\infty P_n \Rightarrow L$ ---

---- $\Sigma^\infty P_n$.

<u>Cor</u> $K_{hC_2} \longrightarrow GW \longrightarrow L$

form a fiber sequence in $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{p}_{\infty}, \mathrm{Sp})$.

<u>Cor</u>. K, L Verdier-localizing

$\Longrightarrow$ GW is so.

(so initial V-localizing functor

under $\Sigma^{\infty} P_n$ ...)

Tate square

$$
\begin{tikzcd}
K_{hC_2} \arrow[r, equal] \arrow[d, dashed] & K_{hC_2} \arrow[d, dashed] \\
GW \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & K^{hC_2} \arrow[d] \\
L \arrow[r] & K^{tC_2} \simeq \rho c K^{hC_2}
\end{tikzcd}
$$