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A biased view of equivariant stable homotopy theory

G : cpt Lie group.

BG s.t.

(∞-)cat $\mathrm{Fun}(BG, Sp)$

is the cat of spectra w/

G-action (Borel G-spectra

$X : BG \to Sp$

$X^{hG} = \lim$

$X_{hG} = \mathrm{colim}$

$X^{tG} = \mathrm{cof}(X_{hG} \to X^{hG})$

Examples : $KR_2 \circlearrowleft C_2$ $E_n \circlearrowleft G$ ← finite

↳ Morava E-theory

· $MU_{\mathbb{R}} = M\mathbb{R} \circlearrowleft C_2$

· Hochschild homology of ring

w/ its Borel S^1-action

Rmk

$KR_2^{tC_2} = 0$

$KR_2^{hC_2} = KO_2$

$M\mathbb{R}^{tC_2} = MO$

Note: BG is rarely a cpt space

$$\Rightarrow \left\{\begin{matrix}\text{Borel} \\ G\text{-Sp}\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{genuine} \\ G\text{-Sp}\end{matrix}\right\}.$$

Example: For any prime p, a genuine

C_p-sp is the data of

① a Borel C_p-sp X

② A sp $\Phi^{C_p} X$ w/ a map

$$\Phi^{C_p} X \longrightarrow X^{tC_p}$$

Rmk: The genuine fixed points X^{C_p} are define as a

$$\begin{tikzcd} X^{C_p} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{hC_p} \arrow[d] \\ \Phi^{C_p} X \arrow[r] & X^{tC_p} \end{tikzcd}$$

Rmk Borel C_p spectra sit fully faithfully inside genuine C_p spectra as those genuine X for which

$$\Phi^{q} X \simeq X^{tC_p}.$$

Example: MU_{C_p} consists of

① MU w/ trivial Borel C_p-action

② $\Phi^{C_p} MU_{C_p} \longrightarrow MU^{tC_2}$

$$\begin{array}{c} \Phi^{C_p} MU_{C_p} \\ \parallel \\ MU \otimes MUP^{\otimes (p-1)} \end{array}$$

<u>Note</u> $\pi_*(MU^{tC_p}) \cong MU_*[[z]][z^{-1}] / [p](z)$

is more complicated than $\pi_*(\Phi^{C_p} MU_{C_p})$

<u>For me</u> To understand Borel G-spectra

one often resolves them by genuine

G-spectra

A few definitions

(Devalapurkar - Hahn - Raksit - Yuan)

Say a G-eqvt sp X is <u>even</u> if,

for all $H \leq G$, and all cplx H

repr V, $\pi^H_{V-1}(X) = 0$.

<u>Example</u> (big thm of Hausmann (et al.))

For cpt abelian G, MU_G is even

w/ understood homotopy groups.

Def. A map $A \to B$ of G-eqvt E_∞-rings is called evenly free if B can be built in the category of A-modules by attaching A-module cells of the form $\Sigma^V A$, V complex

Fix a G-eqvt ring A & suppose $\exists$ an evenly free $A \to B$ w/ B even

Then the G-eqvt even stack for $\Phi^G A$ is that associated to $\pi_*(\Phi^G B^{\otimes_A \bullet +1})$

for A^G is that is that associated

to $\pi_*^G((B^{\otimes A-1}))$

Lemma This does not depend on

the choice of B

Example. If G is cpt abelian

$$\mathbb{S}_G \to MU_G$$

is evenly free w/ even target

Example If $G = \{e\}$, we get $\mathcal{M}_{fg}$

for both $\mathbb{S}^e$ & $\Phi\mathbb{S}$.

This $\mathbb{S} \to MU$ descent, up to p-completion

is related to Cellular $\mathbb{C}$-mot spectra.

Example

$$\mathbb{S}_{C_p} \longrightarrow MU_{C_p}$$

Then stack associated to $\Phi^{C_p}\mathbb{S} = \mathbb{S}$

is the stack associated to

$$\mathbb{S} \longrightarrow \Phi^{C_p} MU = MU \otimes MUP^{\otimes p-1}$$

i.e. $\mathcal{M}_{fg}$ Adss ?

Note Up to p-completions

$\mathbb{S}_{C_p}$ is Borel

$\mathbb{S}^{tC_p} \simeq (MU^{\otimes \bullet+1})^{tC_p}$

there is no clear stack or formal stack attached to π_* of this

Rmk: The stack associated to $\mathbb{S}^{C_p}$ is presented by

$\pi_*^{C_p}((MU_{C_p}^{\otimes \bullet+1})^{\cdot})$

$\mathcal{M}_{C_p}$-equvt fg

WIP of Keita Allen + Lucas Piessevaux

aims to connect descent $\mathbb{S}_{C_p} \to MU_{C_p}$ w/

cellular C_p-eqrt $\mathbb{C}$-mot homotopy theory

based on ideas/WIP of

Devalapurkar + Venkatesh who study the

stacks associated to simple rings (like E_∞)

for complicated connective ring?

Prop The natural map

$$THH(MU) \longrightarrow MU_{C_p} \text{ is evenly}$$

free w/ even target.

Pf Sketch: The map is the Thomification

of a map

$$S(\lambda)_+ \otimes BU_{C_p} \longrightarrow BU_{C_p} \to \mathrm{Pic}(\mathbb{S}_{C_p})$$

Now note, there is a fiber sequence

$$S(\lambda)_+ \otimes BU_{C_p} \to BU_{C_p} \to B^{\lambda} BU_{C_p}$$

$$S(\lambda)_+ \to S^0 \to S^{\lambda}$$

$\Rightarrow MU_{C_p}$, as $THH(MU)$-module, has

the cells of the space $B^{\lambda} BU_{C_p}$.

Lemma: $B^{\lambda} BU_{C_p} = BSU_{C_p}$.

Question: For what V does

$$\Omega^{\infty}\Sigma^{V}\tau_{\geq 0} KU_{G}$$

has repr cells?

What about $\Omega^{\infty}\Sigma^{V} MU_{G}$?

What is the connection to unstable G-eqvt mot. homotopy theory.

The prismatization of complex bordism

G - cpt group.

There is a recipe for constructing

algebro-geometric objects

$$A \longrightarrow B$$

s.t.

① as an A-module, B should have a

cell decomp w/ cells $\Sigma^V A$

where $V \in RU(G)$

② $\pi^G_{V-1}(B)=0$

for all $V \in RU(G)$

Then the G-eqvt even stack

for $\pi^G_*(A)$ is the stack presented

by $\pi^G_*(B^{\otimes A^{\bullet+1}})$

for $\pi_* \Phi^G(A)$ ...

.. $\pi_* \Phi^G(B^{\otimes A^{\bullet}+1})$

Rmk Suppose $G = S^1$

Then S^1-even. stack for $\pi_*((\Phi^{C_p} A)^{hS^1/C_p})$

$$\cdots\quad \pi_*\left(\left(\Phi^{C_p}\left(B^{\otimes_A \cdot +1}\right)\right)^{hS^1/C_p}\right)$$

<u>Example</u>: $G = \{e\}$, $A = \mathbb{S}$

The stack for $\pi_*\mathbb{S}$ is presented by

$\pi_*(MU^{\otimes \cdot +1})$

The stack $\mathcal{M}_{fg} \xrightarrow{\omega} B\mathbb{G}_m$

The ANSS runs

$$\mathcal{H}^a(\mathcal{M}_{fg}, \omega^{\otimes b}) \Rightarrow \pi_{2b-a}(\mathbb{S})$$

Example: If G is a cpt abelian Lie group. the G-eqvt even stack for $\pi_*^G(\mathbb{S}_G)$ is $\mathcal{M}_{G\text{-eqvt fg}}$

Goal Study TC

Fix a p & everything will be p-completed

If R is a non-eqvt $\mathbb{E}_\infty$-ring

$$THH(R) = \operatorname*{colim}_{S^1} R$$

is a S^1 Borel eqvt. $\mathbb{E}_\infty$-ring

$$TC(R) = \text{equalizer}\left(THH(R)^{hS^1} \overset{\text{can}}{\underset{\varphi}{\rightrightarrows}} \left(THH(R)^{tC_p}\right)^{hS^1/C_p} \right)$$

<u>Goal</u> For (suitably nice) $\mathbb{E}_\infty$-rings R, describe a stack associated to $TC(R)$.

The stack is denoted R^{Syn}

There will be a map

$$R^{Syn} \longrightarrow \mathcal{M}_{fg}$$

$$H^*(R^{Syn}, \omega^{\otimes *}) \Longrightarrow \pi_* TC(R)$$

$$\shortparallel$$

$$H^*(R^{Syn}, \mathcal{O}\{*\})$$

Rmk This stack was constructed for every discrete ring R, by Bhatt - Lurie - Drinfeld

Just as

$$TC(R) = \text{equalizer}\left(THH(R)^{hS^1} \rightrightarrows \left(THH(R)^{tC_p}\right)^{hS^1} \right)$$

$$R^{Syn} = \text{coequalizer}\left(R^{\Delta} \rightrightarrows R^{Nyg} \right)$$

First, let us describe R^{Δ}.

This is some stack associated to

$$\left(THH(R)^{tC_p}\right)^{hS^1/C_p}$$

$$= \left(\Phi^{C_p} THH(R)\right)^{hS^1/C_p} \text{ if we view}$$

THH(R) as a Borel S^1-spectrum

<u>Thm</u> Suppose R is an F-smooth discrete ring (e.g. Noetherian regular ring)

Then R^{Δ} is the S^1-eqvt even stack assoc. to $\pi_*\left(\left(\Phi^{C_p} THH\ R\right)^{hS^1/C_p}\right)$ is a formal stack, Complete along a divisor

$R^{\mathcal{HT}}$, the S^1-even stack

assoc. to $\pi_*(\Phi^{C_p} THH(R))$

Example $THH(\mathbb{S}) = \mathbb{S}$ w/ trivial S^1-action

We can compute using $\mathbb{S} \to MU_{S^1}$

$$\begin{array}{ccc} \mathbb{S}^{\mathcal{HT}} & \text{is descent along} & \Phi^{C_p}\mathbb{S} \to \Phi^{C_p} MU_{S^1} \\ \| & & \| \qquad\qquad \| \\ \mathcal{M}_{fg} & & \mathbb{S} \qquad\qquad MU \otimes MUP^{\otimes p-1} \end{array}$$

$\mathbb{S}^{A}$ is presented by Hopf algebroid

"moduli $\{\hat{G}, s, * \to \hat{G}\}$" $\pi_*\left(((MU \otimes MUP^{\otimes p-1})^{\otimes \cdot +1})^{hS^1/C_p}\right)$

equivalently $\mathrm{Tot}\left((MU^{\otimes \cdot +1})^{hS^1} \right)$

<u>Example</u> MU^{HT} can be computed

& $MU^{\mathbb{A}}$

using $THH(MU) \longrightarrow MU_{S^1}$

To understand MU^{HT}, take Φ^{G} to get

$$THH(MU)^{tC_p} \longrightarrow MU \otimes MU_p^{\otimes p-1}$$

$\simeq$ (Rognes, Lunøe-Nielsen)

$MU[SU]$

This descent is the same as the

descent $MU[SU] \longrightarrow SU$

along the augmentation

This is $\pi_* \left(\begin{tikzcd} MU[BSU^{\times 2}] \\ MU[BSU \times BSU] \arrow[u] \arrow[u, shift left] \arrow[u, shift right] \\ MU[BSU] \arrow[u, shift left] \arrow[u, shift right] \end{tikzcd} \right)$

This is the module

$$\left\{ \hat{\mathbb{G}},\ \hat{\mathbb{G}} \times \hat{\mathbb{G}} \to B\mathbb{G}_m,\ \begin{tikzcd} \hat{\mathbb{G}} \arrow[r, bend left, "n(-a)"] \arrow[r, bend right, "0"'] \arrow[r, phantom, "\Updownarrow"] & B\mathbb{G}_m \end{tikzcd} \right.$$

trivial $B\mathbb{G}_m$-valued

sym 2-cocycle

Thm MU^{Δ}, the algebro-geometric object computing $(THH(MU)^{tC_p})^{hS^1/C_p}$

is

$$\left\{ \hat{G},\ \underset{\substack{\text{trivial}\\ \text{sym}\\ \text{2-cocycle}}}{s: * \longrightarrow \hat{G},\ \hat{G}\times\hat{G} \xrightarrow{g} BG_m},\ \underset{\substack{\text{res. of}\\ g \text{ along the}\\ \text{section}}}{\begin{tikzcd} \hat{G} \arrow[r, bend left, "\chi(-e)"] \arrow[r, bend right, ""'] \arrow[r, phantom, "\Updownarrow"] & BG_m \end{tikzcd}} \right\}$$

Conjecture. The motivic spectral sequence

from the cohomology of this to

$\pi_*(THH(MU)^{tC_p})^{hS^1/C_p}$ collapse at E_2

Given an $\mathbb{E}_\infty$-ring R

R^{Nyg} is a stack assoc. to

$THH(R)^{hS^1}$

Def R^{Nyg} is the completion at v_0

of the S^1-eqvt even stack computing

$\pi_*^{S^1} THH(R)$.

Roughly, look at $\left(\pi_*^{C_p} THH(R)\right)^{hS^1/C_p}$

$\mathbb{S}^{Nyg}$

Moduli of S^1-eqvt fgs completed at v_0

$\Theta : \mathbb{Z} \to \hat{G}$

w/ $\Theta(p)$ is a nbhd of $\Theta(0)$.

$$MU^{Nygg} = \{ \hat{G},\ \Theta : \mathbb{Z} \to \hat{G},\ \hat{G} \times \hat{G} \to B\mathbb{G}_m,$$

$$\hat{G} \begin{tikzcd} {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Updownarrow"] & {} \end{tikzcd} B\mathbb{G}_m$$

S^1-eqvt fg complete at v_0