Emanuele Dotto

§1) Milnor K-theory & Milnor's conjecture

Let k be a field (eventually of char 2)

and let $K_*(k)$ be the algebraic K-group of k.

$$K_0(k) \xrightarrow[\simeq]{\dim} \mathbb{Z}.$$

$$K_1(k) \xrightarrow[\simeq]{\det} k^\times.$$

$$K_2(k) \cong k^\times \otimes k^\times / \langle a \otimes (a-1) \mid a \neq 0, 1 \rangle$$

Def [Milnor]

$$K_*^M(k) := T_*^{\otimes}(k^\times, \cdot) \big/ \langle a \otimes (1-a) \mid a \neq 0, 1 \rangle$$

graded ring w/

map of graded rings

$$K_*^M(k) \longrightarrow K_*(k)$$

(not an iso. in general)

Milnor's conjecture relates

$K_*^M(k)/2$ w/ symmetric form

<u>Def</u> A non-degenerated symmetric form over k consists of a pair (V, φ) of a finite dim space V and an iso. .

$$\varphi : V \xrightarrow{\sim} V^* \quad \text{s.t.} \quad \varphi^* \varphi = ev.$$

A Lagrangian of (V,φ) consists of a vector subspace $L \subseteq V$ s.t.

- $\varphi|_L = 0$ &
- $$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, hook] & V \arrow[rr] \arrow[dr, "\varphi"'] & & L^* \arrow[r] & 0 \\ & & & V^* \arrow[ur] & & \end{tikzcd}$$ is exact

Def) The Witt group of k is

$$W^s(k) = \underbrace{\left(\{(V,\varphi)\}/_{\cong} \,,\ \oplus \right)}_{\text{monoid}} \Big/ \langle (V,\varphi) \text{ w/ Lagrangian} \rangle$$

This is in fact a group

$$-[V, \varphi] = [V, -\varphi].$$

and a ring by $\otimes$

$$I = \ker \left(W^s(k) \longrightarrow \mathbb{Z}/2 \right)$$
$$[V, \varphi] \longmapsto \dim V$$

<u>Conjecture</u> [Milnor &c]

$$K_*^M(k)/2 \xrightarrow{\sim} I^*/I^{*+n}$$
$$a \in k^\times \longmapsto (1 - [k, a])$$

$\underline{Q}$ What about something else than K?

$\underline{\S 2)}$ THH and Milnor's conj.

Bökstedt:

$$THH(k) = k \underset{k \otimes k}{\otimes} k \quad \text{a ring spectrum}$$
$$\mathbb{S}$$

$\pi_0(THH(k)) \simeq k$

$\pi_1(THH(k)) \simeq \Omega^1_k$

$\Omega^*_k := \Lambda^* \Omega^1_k$. free cdga on k w/ $(dx)^2 = 0$.

Then there is a com. diagram

$$\begin{tikzcd} a \arrow[d, mapsto] & K_*^M(k) \arrow[r] \arrow[d] & K_*(k) \arrow[d, "h"] \\ \frac{da}{a} & \Omega_k^* \arrow[r] & \pi_* THH(k) \end{tikzcd}$$

let k be of char 2

$S \subset k$ subfield of squares

$$J = \ker(\mu : k \otimes_S k \longrightarrow k)$$

Prop $\quad \Omega_k^* \xrightarrow{\sim} J^*/J^{*+1}$

Let's try to relate this results

- <u>K-theory</u> There is a preferred refinement of $K(k)$ to a $\mathbb{Z}/2$-spectrum

s.t.

$$
\begin{array}{ccc}
W^s(k) & \cong & \pi_0\left(K(k)^{\Phi\mathbb{Z}/2}\right) \\
\Big\downarrow{\scriptstyle \dim} & & \Big\downarrow{\scriptstyle \mathrm{res}^{\Phi\mathbb{Z}/2}} \\
\mathbb{Z}/2 & \cong & \pi_0 K(k)/2
\end{array}
$$

<u>Thm</u> $K^M_*(k)/2 \cong I^*/I^{*+1}$

where $I = \ker(\mathrm{res}^{\mathbb{Z}/2})$

• THH There is a preferred refinement of $THH(k)$ to a $\mathbb{Z}/2$ - spectrum [Hesselholt - Madsen]

s.t.

$$\begin{tikzcd} \pi_0 \left(THH(k)^{\Phi \mathbb{Z}/2} \right) \arrow[d, "res^{\Phi \mathbb{Z}/2}"'] \arrow[r, phantom, "\cong"] & k \otimes_{S} k \arrow[d, "\mu"] \\ \pi_0 THH(k)/2 \arrow[r, phantom, "\cong"] & k \end{tikzcd}$$

Thus $\Omega_k^* \cong J^*/J^{*+1}$, $J = \ker res^{\Phi \mathbb{Z}/2}$.

§3) The de Rham-Witt cplx and Madsen's Conjecture

Bokstedt - Hsiang - Madsen

$THH(k)$ is refined to a C_n-spectrum

$\forall n \geq 1$

$$TR_{*}^{n+1}(k;p) := \pi_0\left(THH(k)^{C_{p^n}}\right) \quad \forall n \geq 0$$

$$TR_0^{n+1}(k;p) \cong W_{\langle p \rangle}(k) \sim$$ (n+1)-truncated p-typical Witt vectors

$$\left(W_{\langle p \rangle}(\mathbb{F}_p) \cong \mathbb{Z}/p^{n+1}\right)$$

$$TR_1^{n+1}(k;p) \cong \Omega^1 W_{\langle p^n \rangle}(k)/\sim \quad ?$$

There are operators on these groups

$$TR_*^n(k;p) \underset{R}{\overset{V}{\underset{F}{\rightleftarrows}}} TR_*^{n+1}(k;p)$$

& a differential s.t. $\underbrace{\_\ \_\ \_}_{\text{a Witt complex}}$

<u>Def</u> [Hesselholt - Madsen]

Bloch - Deligne - Illusie

$W_\bullet \Omega^*_k$ is the initial object in the category of objects w/ the struct. of $TR^{\bullet+1}_*(k;p)$

There is a preferred refinement of $TR^{n+1}(k;p)$ to a $\mathbb{Z}/2$ - spectrum [Høgenhaven]

let $J_\bullet := \cancel{\ker}\left(res^{\Phi \mathbb{Z}/2} : \pi_0 (TR^\bullet(k;2))^{\mathbb{Z}/2} \to \pi_0 TR^\bullet(k,2) \right)$

$W_{\langle 2 \rangle}(k)/2$

Thm [D.]

$$W.\Omega^*_k/2 \cong J^*_\cdot / J^{*+1}_\cdot .$$

Rmk

- If char $k \neq 2$, then

$$\Omega W_k/2 = 0 \quad \& \quad TR(k,2)^{\Phi \mathbb{Z}/2} = 0.$$

- The proof computes $J_\cdot$ and identifies it with $W\Omega/2$ algebraically

- Then is also a version for TC.

  but it is equiv. to the Milnor conj..