Mackey functors in equivariant homotopy theory

A Handbook to Homotopy Theory Ch. 17

Webb A Guide to Mackey Functors

G finite group

Top^G : Cat of G-spaces & G-maps

w.e. in Top^G :

w.e. after taking $(-)^H$ $\forall H < G$.

G-CW cplx :

$$\begin{tikzcd} G/H \times S^{n-1} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & G/H \times D^n \arrow[d] & \cdots \\ X_{n-1} \arrow[r] & X_n & \end{tikzcd}$$

Rmk. Zero cells — G-orbits

Equivariant Whitehead Theorem

- ...

homotopy groups

$$\pi_n^H = \pi_n \circ (-)^H.$$

really htpy classes of pointed maps!

use Top_*^G — objects have a

G-fixed basepoint

Equivariant cohomology

Bredon cohomology $H_G^*(-)$.

requires a coeff. system.

$$H_G^*(G/K) \longrightarrow H_G^*(G/J)$$

$$M : \mathrm{Orb}_G^{op} \longrightarrow \mathrm{Ab}$$

[better $\quad m : \mathrm{Fin}_G^{op} \to \mathrm{Ab}, \quad \amalg \longmapsto \oplus$]

$$H_G^*(X; m)$$

V real ortho. G-repr.

$\rightsquigarrow S^V$ repr. sphere

$\Sigma^V X$

to incorporate $\rightsquigarrow$ $RO(G)$-graded cohom.

$RO(G) = \mathbb{Z}\langle$ irred. ortho. real G-repr$\rangle$

<u>Thm</u> (Lewis, May, McClure 1981)

Bredon cohomology extends to an $RO(G)$-graded cohom. theory iff the coeff. system in extends to a Mackey functor.

$\alpha \in RO(G)$ virtual rep
$\underline{m}$ - Mackey functor
$\Big\} \Rightarrow H^{\alpha}_{G}(X; \underline{m})$

w/ suspension iso.

$$\tilde{H}^{\alpha}_{G}(X; \underline{m}) \cong \tilde{H}^{\alpha+V}_{G}(\Sigma^{V} X; \underline{m})$$

<u>Brown representability</u>:

$H^{*}_{G}(-, \underline{m})$ is represented by a genuine equivariant G-spectra

EM spectra $H\underline{m} \in Sp^G$.

"extends" b/c $\alpha = \mathbb{R}^n_{triv}$

$\Rightarrow H^\alpha(X; \underline{m})$

$\cong H^n(X; m)$ m

<u>Def</u> A Mackey functor $\underline{m}$ is

$$\begin{cases} \underline{m}^* : \mathrm{Fin}_G^{op} \to \mathrm{Ab} \\ \underline{m}_* : \mathrm{Fin}_G \to \mathrm{Ab} \end{cases}$$

s.t. 1) $\underline{m}^* = \underline{m}_*$ on objects

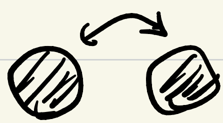
2) $\underline{m}_*, \underline{m}^*$ commutes w/ finite coprod.

3) "push – pull"

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner"] & R \arrow[d, "i"] \\ S \arrow[r, "h"'] & T \end{tikzcd} \rightsquigarrow \begin{tikzcd} \underline{m}(P) \arrow[r, "f_*"] & \underline{m}(R) \\ \underline{m}(S) \arrow[u, "g^*"] \arrow[r, "h_*"'] \arrow[ur, phantom, "\circ"] & \underline{m}(T) \arrow[u, "i^*"'] \end{tikzcd}$$

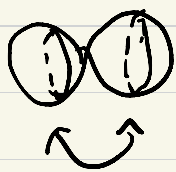
$G = C_2$

Orb_G



$\underline{\text{Ex}}$ $\cdot$ pt $\longleftarrow$ $C_2/e \times D^n$

$\downarrow$



$C_2/e \longrightarrow pt = C_2/C_2$

irreps $\mathbb{R}_{triv}$, $\mathbb{R}_{sgn} = \sigma$.

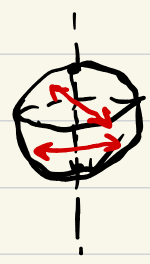
$$V \cong \mathbb{R}^{p,q} = \mathbb{R}_{triv}^{p-q} \oplus \mathbb{R}_{sgn}^{q}$$

$S^V = S^{p,q}$ top dim $= p$.

twisted dim $= q$.

$S^{1,0}$ $\quad$ $S^{1,1}$ $\quad$ $S^{2,2}$

S^1 $\quad$ S^σ

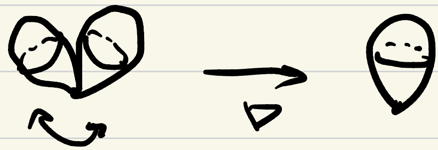


<u>Ex</u> Find C_2-CW struct. on

$S^{1,1}$, $S^{2,2}$, S^1_{a+}, S^2_{a+}

anti-podal

$C_2 \xrightarrow{p} pt \quad \xrightarrow{\text{Suspend}} \quad C_{2+} \wedge S^V \to S^V$



$C_2 \dashleftarrow pt$ no maps $\quad C_{2+} \wedge S^V \leftarrow S^V$

but stably there are!

Pontrjagin - Thom Collapse

C_2-Mackey functors

$$\begin{tikzcd} C_2/C_2 \\ C_2/e \arrow[u, "p"] \arrow[loop below, "t"] \end{tikzcd} \qquad \underline{m}: \quad \begin{tikzcd} \underline{m}(C_2/C_2) \arrow[d, shift right, "\mathrm{res}^{C_2}_e := p^*"'] \\ \underline{m}(C_2/e) \arrow[u, shift right, "p_* = \mathrm{tr}^{C_2}_e"'] \arrow[loop below, "t^* = t_* =: t"] \end{tikzcd}$$

(specific to C_2)

w/ relations

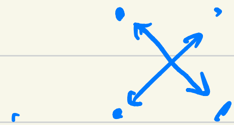
$$p_* t = p_*$$

$$t p^* = p^*$$

$$t^2 = \mathrm{id}$$

$$p^* p_* = \mathrm{id} + t$$

from push-pull



$$\begin{tikzcd} C_2/e \arrow[d, "p"] \arrow[dr, dashed] & C_2/e \sqcup C_2/e \arrow[l, "(\mathrm{id},\mathrm{id})"'] \arrow[d, "(\mathrm{id},t)"] \\ C_2/C_2 & C_2/e \arrow[l, "p"] \end{tikzcd}$$

Ex $\mathbb{F}_2$

constant

$$\begin{tikzcd} \mathbb{F}_2 \arrow[d, shift right, "1"'] \\ \mathbb{F}_2 \arrow[u, shift right, "0"'] \arrow[loop below, "1"] \end{tikzcd}$$

$\mathbb{Z}$

$$\begin{tikzcd} \mathbb{Z} \arrow[d, shift right, "1"'] \\ \mathbb{Z} \arrow[u, shift right, "2"'] \arrow[loop below, "1"] \end{tikzcd}$$

RU

$$\begin{tikzcd} RU(C_2) \arrow[d, shift right, "\mathrm{res}"'] \\ RU(e) \arrow[u, shift right, "\mathrm{ind}"'] \arrow[loop below, "?\ \mathrm{id}"] \end{tikzcd}$$

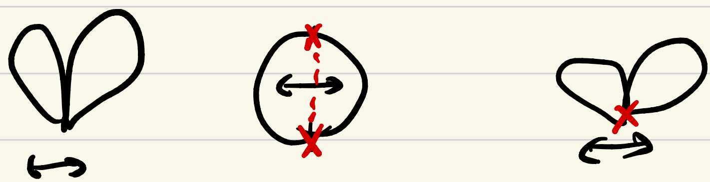
$$\begin{tikzcd}
\mathbb{F}_2 \arrow[d, shift right, "\Delta"'] \\
\mathbb{F}_2[C_2] \arrow[u, shift right, "\nabla"'] \arrow[loop below]
\end{tikzcd} =: F
\qquad\qquad
\begin{tikzcd}
V^{C_2} \arrow[d, hook, shift right] \\
V \arrow[u, shift right, "\mathrm{fpt}"'] \arrow[loop below, "C_2"]
\end{tikzcd}$$

$$C_2 \longrightarrow pt$$

push-pull :

$$p^* \qquad\qquad p_*$$

$$\begin{tikzcd}
C_{2+} \wedge S^\sigma \arrow[r, "p \wedge id"] \arrow[rr, bend right, "id + t"'] & S^\sigma \arrow[r, "\text{pinch}"] & C_{2+} \wedge S^\sigma
\end{tikzcd}$$



Back to G - finite

$RO(G)$- graded homotopy

$$\pi_V^G(X) = [S^V, X]^G$$

$$\underline{\pi}_V^G(X)(G/H) = [G_+ \wedge_H S^V, X]^G. \quad \text{Mackey functor}$$

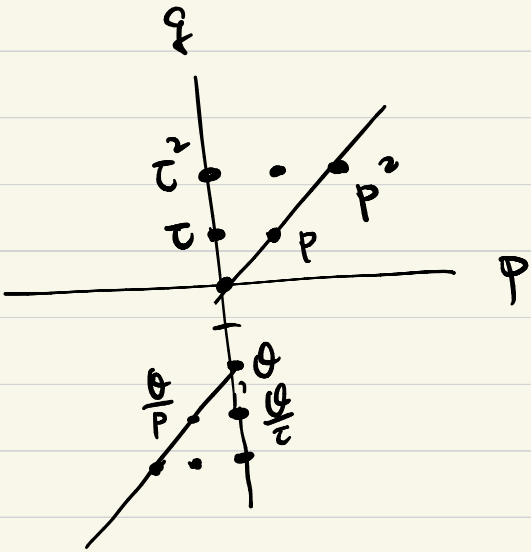
$RO(G)$-gr cohom rep'd by $H\underline{M}$

characterised $\underline{\pi}_n(H\underline{M}) = \begin{cases} \underline{M}, & n=0 \\ 0, & \text{o/w} \end{cases}$

$\underline{\pi}_V^G(H\underline{M}) =$ often nonzero

Ex $\pi^{C_2}_{*,*}(H\underline{\mathbb{F}}_2) \cong H^{*,*}(pt, \underline{\mathbb{F}}_2) = \mathbb{M}_2$

$\bullet = \mathbb{F}_2$



non-Noetherian ring

$\tau = u_\sigma = t_\sigma$

$\rho = a_\sigma$

$\theta^2 = 0$

lots of richer struct. in the equivariant world

<u>classical</u>	<u>equivariant</u>
abelian groups $\rightsquigarrow$	Mackey functor
$\otimes$	$\square$
(comm) ring	(comm) Green functor (Mackey ring)
	Tambara functor

$\underline{M}: BG^{op} \to Ab$

$\wr$

Burnside cat

Modules of Equivariant EM spectra

classical homotopy

$H^*(X, \mathbb{F}_p)$ - graded $\mathbb{F}_p$ v.s.

$\bigoplus$ of shifts of $\mathbb{F}_p$

splitting is reflected stably

$Y \in H\mathbb{F}_p$ - mod $\Rightarrow$ Y splits

as $\bigvee$ of Σ of $H\mathbb{F}_p$.

$$Y \simeq \bigvee_{i \in I} \Sigma^{n_i} H\mathbb{F}_p.$$

Thm (Hopkins – Smith 1998)

R ring spectrum w/ $\pi_* R$ is a graded field

$Y \in R\text{-Mod} \Rightarrow Y$ splits as $\bigvee$ of suspension of R

Ex

- $R = H\mathbb{F}_p$
- $R = KU_{\mathbb{Q}}$ $\quad \pi_* KU_{\mathbb{Q}} \simeq \mathbb{Q}[\beta^{\pm 1}]$, $\quad |\beta| = 2$
- $R = K(n)$ $\quad \pi_* K(n) \cong \mathbb{F}_p[v_n^{\pm 1}]$, $\quad |v_n| = 2p^n - 2$

Equivariant homotopy

$\underline{\underline{G = C_2}}$

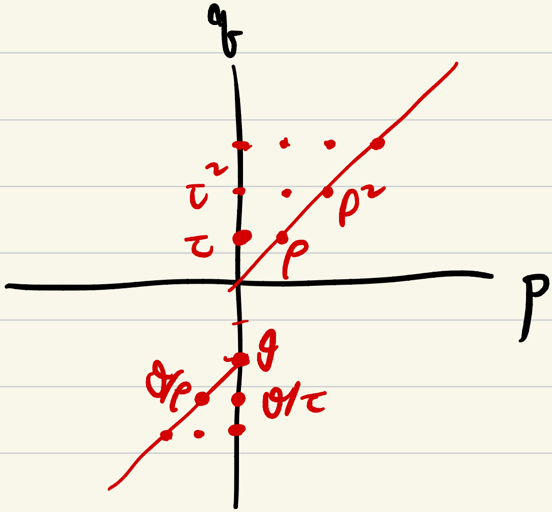
$V \cong \mathbb{R}^{p,q} = \mathbb{R}^{p-q}_{triv} \oplus \mathbb{R}^q_{sgn}$

$S^V = S^{p,q}$

$RO(C_2)$-graded cohomology is bigraded

$H^{*,*}_{C_2}(-, \underline{\mathbb{F}_2})$ rep'd by EM spectrum $\underline{H\mathbb{F}_2}$

$\mathbb{M}_2 = H^{*,*}_{C_2}(pt, \underline{\mathbb{F}_2}) \cong \pi^{C_2}_{*,*} H\underline{\mathbb{F}_2}$



$\bullet = \mathbb{F}_2$

$\theta^2 = 0$ $\quad \tau\theta = \rho\theta = 0$

$\tau = u_\sigma$

$\rho = a_\sigma$

not a field

non-Noetherian

<u>Thm</u> (M 2018)

X - finite C_2-CW cpx

$\Rightarrow H^{*,*}_{C_2}(X, \underline{\mathbb{F}_2})$ is a $\oplus$ of (rep) shifts of $\mathbb{M}_2 = H^{*,*}_{C_2}(pt, \underline{\mathbb{F}_2})$

and $H^{*,*}_{C_2}(S^n_a; \mathbb{F}_2)$

$n \geq 0$

Splitting is reflected stably

<u>Thm</u> (M. 2019)

Y is a finite C_2-CW spectrum

$\Rightarrow Y \wedge H\underline{\mathbb{F}}_2$ splits as a $\bigvee$ of (rep)

Σ of

$H\underline{\mathbb{F}}_2$ and $(S^n_a)_+ \wedge H\underline{\mathbb{F}}_2$

Proofs used arguments in bigraded homotopy.

Some ingredients: • $\pi^{C_2}_{*,*} H\underline{\mathbb{F}}_2$ is self-injective

• Toda bracket $\langle \tau, \theta, \rho \rangle = 1$

more generally? Need one more piece

Thm (Dugger - Hazel - M. 2024)

$Z \in H\underline{\mathbb{F}}_2\text{-Mod}^C \Rightarrow Z$ splits as

$\bigvee$ of (rep) Σ of

$H\underline{\mathbb{F}}_2$, $(S^n_a)_+ \wedge H\underline{\mathbb{F}}_2$, and $\mathrm{cof}(\tau^m)$
$n \geq 0$ $\quad m \geq 1$

i.e. $\exists$ families of (iso classes of) indecomposables

$H\underline{\mathbb{F}}_2$ not a field, not a PID

but module theory looks like a PID

Proof used very different techniques

<u>Thm</u> (Schwede - Shipley 2003)

$$H\underline{\mathbb{F}}_2\text{-Mod} \underset{Q.E}{\simeq} Ch(\underline{\mathbb{F}}_2)$$

DHM: described $D^{perf}(\underline{\mathbb{F}}_2)$

- classified $\underline{\mathbb{F}}_2$-modules
  - 5 indecomposables
  - 2 projective

$$\begin{tikzcd} \mathbb{F}_2 \arrow[d, shift right, "c"'] \\ \mathbb{F}_2 \arrow[u, shift right, "0"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{F}_2 \arrow[d, shift right, "\Delta"'] \\ \mathbb{F}_2[C_2] \arrow[u, shift right, "\nabla"'] \end{tikzcd}$$

H $\qquad$ F

- classified perfect complexes of $\underline{\mathbb{F}}_2$-mods via a change of basis algorithms

<u>Thm</u> (DHM 2024)

Up to q-iso any perfect complex splits as a $\oplus$ of shifts of "single strands"

$$F \xrightarrow{1+t} F \xrightarrow{1+t} \cdots \to F \to F \qquad (S^{n}_{n})_{+} \wedge H\underline{\mathbb{F}}_2$$

$$F \to F \to \cdots \to F \to F \to H \qquad S^{n,n} \wedge H\underline{\mathbb{F}}_2$$

rep suspensions

$$H \to F \to F \to \cdots \to F \to F \qquad S^{-n,-n} \wedge H\underline{\mathbb{F}}_2$$

$$H \to F \to F \to \cdots \to F \to F \to H \qquad \mathrm{cof}(\tau^{n})$$

some trivial stuff

$G = C_p$ p odd prime

$H\underline{\mathbb{F}}_p$ represents $RO(C_p)$-graded

cohom $H^{*}_{C_p}(-; \mathbb{F}_p)$

$\pi_*^{C_p} H\underline{\mathbb{F}}_p$ is self-injective, Toda bracket,

Some indecomposables.

$H\underline{\mathbb{F}}_p$, $C_{p+} \wedge H\underline{\mathbb{F}}_p$, $(S^{2n+1}_{free})_+ \wedge H\underline{\mathbb{F}}_p$,

$\underline{E}B \wedge H\underline{\mathbb{F}}_p$

<u>Thm</u> (Grevstad - M. in progress)

The classification of cpt $H\underline{\mathbb{F}}_p$-modules is wild.

<u>morally</u> : impossible to describe all the indecomposables

- such a list would contain every indecomposable module of every finite dim. $\mathbb{F}_p$-alg.

<u>Ex</u> Modular rep thy

$\mathbb{F}_p[C_p \times C_p]$ is wild unless $p=2$

(e.g. $\mathbb{F}_2\langle x, y\rangle$ is wild)

R k-alg k: field

R has <u>wild rep type</u> if $\exists$

$k\langle x, y\rangle\text{-Mod} \longrightarrow R\text{-Mod}$

repr. embedding : i.e. reflecting

iso. and preserving indecomposability

S - f. d. k-alg $S \cong k\langle x_1, \dots, x_n \rangle / \text{rel}$

$S\text{-Mod} \longrightarrow k\langle x, y \rangle\text{-Mod} \longrightarrow R\text{-Mod}$

G<u>M</u> : proof uses a new connection

between modular rep thy and

eqvt htpy

We use quiver reprs to show

$\mathbb{F}_p$ is '

- repr. finite
  - ↪ finitely many indecomps
    - 2 projectives
- derived wild
  - ↪ $D^{perf}(\underline{\mathbb{F}}_p)$ is wild

key idea: A Mackey functor is a rep of a bound quiver w/ relations (Webb)

Rmk For $G = C_2$, an $\mathbb{F}_2$-mod "gentle alg"

is a rep of $\begin{tikzcd} \bullet \arrow[r, bend left, "a"] & \bullet \arrow[l, bend left, "b"] \end{tikzcd}$ $ab = 0$

$(ba \neq 0)$

In fact, we can show the "wildness"

comes from spaces

Thm (Grevstad - M. in progress)

There is no struct. thm for

$RO(C_p)$-graded cohom. w/ $\underline{\mathbb{F}_p}$-coeff

for $p \geq 7$.

[Conj. $p = 3, 5$ also wild, TBD]

Thm (G-M in progress)

The classification of cpt $H\mathbb{F}_p$-mods is wild if

$$\left.\begin{array}{l} \bullet\ G = C_{p^n} \\ \bullet\ G = C_p \times \cdots \times C_p \end{array}\right\} \text{if } |G|>2$$

any prime

go back to modular rep thy

maybe to see sth new