Simplicial Nerve, S and Cat_∞

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1 Preliminary

Note that we have monoidal functors

$$\begin{aligned} c &\colon \mathsf{Set} \to \mathsf{sSet}, \\ \pi_0 &\colon \mathsf{sSet} \to \mathsf{Set}, \\ \mathrm{ev}_0 &\colon \mathsf{sSet} \to \mathsf{Set}, \end{aligned}$$

where c is the constant functor, ev_0 is the functor given by $\mathsf{sSet} \ni X \mapsto X_0$.

We also remark that we have adjoint relations

$$\pi_0 \dashv c \dashv \mathrm{ev}_0 .$$

Denoting

$$\begin{aligned} c = c_* &\colon \mathsf{Cat} \to \mathsf{Cat}_\Delta, \\ \pi = (\pi_0)_* &\colon \mathsf{Cat}_\Delta \to \mathsf{Cat}, \\ u = (\mathrm{ev}_0)_* &\colon \mathsf{Cat}_\Delta \to \mathsf{Cat}, \end{aligned}$$

we then obtain adjoint relations

$$\pi \dashv c \dashv u.$$

2 Simplicial Nerve

In this section we define the so-called simplicial nerve (also named homotopy-coherent nerve), denoted by $\mathrm{N} \colon \mathsf{Cat}_\Delta \to \mathsf{sSet}$.

Beforehand, recall a lemma as follows.

Lemma 1. *Suppose that $\mathcal{C}$ and $\mathcal{D}$ are locally small categories, and that $\mathcal{D}$ admitting all small limits. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor.*

Define $F^* \colon \mathcal{D} \to \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$, which assigns each object $d \in \mathcal{D}$ the presheaf $\mathrm{Hom}_{\mathcal{D}}(F(-), d)$.

Denote by よ $\colon \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}}$ the Yoneda embedding.

Denote by $F_! \colon \mathsf{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{D}$ the left Kan extension of F along よ.

Then, we have adjoint pairs $F_! \dashv F^*$. In particular, $F_!$ preserves colimits.

We give some examples.

Example 1. Let $\mathcal{C} = \Delta$, $\mathcal{D} = \mathsf{Cat}$, and $F\colon \Delta \hookrightarrow \mathsf{Cat}$ the fully faithful functor that embeds Δ as a full subcategory of Cat.

Then $F^* = \mathrm{N}$ the ordinary nerve functor, $F_! = h$ the homotopy functor.

Example 2. Again let $\mathcal{C} = \Delta$, $\mathcal{D} = \mathsf{Top}$, and $F\colon \Delta \to \mathsf{Top}$ sending each object $[n] \in \Delta$ to the geometric n-simplex $|\Delta^n|$.

Then $F^* = \mathrm{Sing}$ the singular complex functor, $F_! = |\cdot|$ the geometrical realisation.

Now, we would like to define the so-called simplicial nerve functor $\mathrm{N}\colon \mathsf{Cat}_\Delta \to \mathsf{sSet}$. To do so, we first construct a functor $\Delta \to \mathsf{Cat}_\Delta$, which serves as F in Lemma 1, and then we define N to be F^*. Besides, the biproduct $F_!$ is denoted $\mathfrak{C}\colon \mathsf{sSet} \to \mathsf{Cat}_\Delta$, which is called the Joyal rigidification functor.

2.1 Construction of the functor $\Delta \to \mathsf{Cat}_\Delta$

Denote by $\mathsf{Lin.or.Set}$ the category of finite linearly ordered sets and order-preserving morphisms.

We are about to define a functor (by some abuse of notation) $\mathfrak{C}\colon \mathsf{Lin.or.Set} \to \mathsf{Cat}_\Delta$, then restrict it to the subcategory Δ of $\mathsf{Lin.or.Set}$, so as to obtain the desired functor $F = \mathfrak{C}\colon \Delta \to \mathsf{Cat}_\Delta$.

To do so, we first do some preparatory work.

For any object $J \in \mathsf{Lin.or.Set}$, and any element i, j in J such that $i \leqslant j$, define

$$P_{i,j} \coloneqq \{I \subset [i,j] \subset J \mid i, j \in I\}.$$

We order $P_{i,j}$ by using the inclusion relation $\subset$, then $P_{i,j}$ can be viewed canonically as a category.

Sometimes we also use the notation $P^J_{i,j}$ for $P_{i,j}$ if we desire to emphasize the whole set J.

Now, for each $J \in \mathsf{Lin.or.Set}$, we assign a simplicially enriched category $\mathfrak{C}[\Delta^J]$, whose object-set is exactly J, and whose hom-sets are given by

$$\mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i,j) \coloneqq \begin{cases} \mathrm{N}\left(P_{i,j}\right), & \text{if } i \leqslant j; \\ \varnothing, & \text{otherwise,} \end{cases}$$

where $i, j \in J$.

Note that for $i \leqslant j \leqslant k$, we have a map $P_{j,k} \times P_{i,j} \to P_{i,k}$ which sends (I', I) to $I \cup I'$. By taking the nerve functor, we obtain a map

$$\mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(j,k) \times \mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i,j) \to \mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i,k),$$

which we define as the composition map in $\mathfrak{C}[\Delta^J]$.

One can check that in this way we indeed defined a simplicially enriched category $\mathfrak{C}[\Delta^J]$.

We are now in position to define the functor $\mathfrak{C}\mathsf{Lin.or.Set} \to \mathsf{Cat}_\Delta$.

Indeed, at the level of objects, we assign to each $J \in \mathsf{Lin.or.Set}$ the simplicially enriched category $\mathfrak{C}[\Delta^J]$.

At the level of morphism, to each order-preserving map $f\colon J \to J'$, we define a map $P^J_{i,j} \to P^{J'}_{f(i),f(j)}$, given by $P^J_{i,j} \ni I \mapsto f(I) \in P^{J'}_{f(i),f(j)}$. Therefore, after taking the nerve functor, we obtain a morphism between simplicial sets

$$f_*\colon \mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i,j) \to \mathrm{Hom}_{\mathfrak{C}[\Delta^{J'}]}(f(i), f(j)).$$

We now define $\mathfrak{C}(f)$ to be f_*.

It can be checked that $\mathfrak{C}$ defined in this way is indeed a functor.

2.2 N and $\mathfrak{C}$

Letting F be the restriction of $\mathfrak{C}$ to Δ, and using Lemma 1, we obtain adjoint functors $\mathrm{N} = F^* \colon \mathsf{Cat}_\Delta \to \mathsf{sSet}$ and $\mathfrak{C} = F_! \colon \mathsf{sSet} \to \mathsf{Cat}_\Delta$. We also note that $\mathfrak{C} \colon \mathsf{sSet} \to \mathsf{Cat}_\Delta$ preserves all colimits.

2.3 Explore $\mathfrak{C}[\Delta^n]$ (and Dwyer–Kan–Bergner model structure)

In this subsection we explore the simplicially enriched category $\mathfrak{C}[\Delta^n]$.

Note that for any $J \in \mathsf{Lin.or.Set}$, one can always find an isomorphism $J \cong [n]$. Thus $\mathfrak{C}[\Delta^J] \cong \mathfrak{C}[\Delta^n]$.

First of all, the objects of $\mathfrak{C}[\Delta^n]$ are exactly elements of $[n] = \{0 < 1 < \cdots < n\}$.

Next, for any $i \leqslant j$, we need to study the category (or ordered set) $P_{i,j}$. If $i = j$, clearly $P_{i,j}$ consists of a single element.

Now suppose $i < j$. Consider a bijection $P_{i,j} \to [1]^{j-i-1}$ such that

$$P_{i,j} \ni I \mapsto \left(\chi_{i+1 \in I}, \cdots, \chi_{j-1 \in I}\right),$$

where for $i < k < j$,

$$\chi_{k \in I} := \begin{cases} 0, & \text{if } k \notin I; \\ 1, & \text{if } k \in I. \end{cases}$$

Thus, we see that, for $i < j$, $\mathrm{N}(P_{i,j}) \cong (\Delta^1)^{j-i-1}$; and $\mathrm{N}(P_{i,i}) \cong \Delta^0$.

As a corollary, it can be checked easily that

Corollary 1. *There exists a canonical isomorphism $\pi(\mathfrak{C}[\Delta^n]) \cong [n]$ which is identity on objects.*

Here, let us add some remarks.

Indeed, the category Cat_Δ can be given a standard model category structure, which is called the Dwyer–Kan–Bergner structure.

To be explicit, we can define the weak equivalence on Cat_Δ as simplicially enriched functors $F \colon \mathcal{C} \to \mathcal{D}$ between arbitrary simplicially enriched categories $\mathcal{C}$ and $\mathcal{D}$ such that

- on the level of morphisms, the map $F \colon \operatorname{Hom}_{\mathcal{C}}(x, y) \to \operatorname{Hom}_{\mathcal{D}}(F(x), F(y))$ is a weak equivalence with respect to the standard model category structure on sSet, that is, it is a weak equivalence after applying geometrical realisation;
- the induced map $\pi(\mathcal{C}) \to \pi(\mathcal{D})$ is essentially surjective.

The weak equivalence defined above is also known as Dwyer–Kan equivalence.

By the way, we define the notion of fibration on Cat_Δ, which is actually of no significance in our situtaion.

The fibrations on Cat_Δ is given by $F \colon \mathcal{C} \to \mathcal{D}$ such that

- on the level of morphisms, the map $F\colon \mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{D}}(F(x),F(y))$ is fibration with respect to the standard model category structure on sSet, that is, it is a Kan fibration;
- the induced map $\pi(\mathcal{C}) \to \pi(\mathcal{D})$ is an isofibration.

Remark 1. We see immediately that the fibrant objects for the Dwyer–Kan–Bergner model are exactly those Kan-enriched categories.

Remark 2. Denote by $\mathcal{H}$ the homotopy category of sSet (or equivalently, that of Kan). We observe that $\mathcal{H}$ is a monoidal category. Also, note that $\pi\colon \mathsf{Cat}_\Delta \to \mathsf{Cat}$ lifts to $\pi\colon \mathsf{Cat}_\Delta \to \mathsf{Cat}_{\mathcal{H}}$ (the latter the category of $\mathcal{H}$-enriched categories). Then, we see that the definition of Dwyer–Kan equivalence is the same as $\mathcal{H}$-enriched equivalence.

Remark 3. One can show that, the homotopy functor $h\colon \mathsf{sSet} \to \mathsf{Cat}$ (i.e. the left adjoint of the nerve functor $\mathrm{N}\colon \mathsf{Cat} \to \mathsf{sSet}$) coincides with the compostion $\pi \circ \mathfrak{C}$. Thus, according to the previous remark, h can be lifted to $h\colon \mathsf{sSet} \to \mathsf{Cat}_{\mathcal{H}}$.

As a sequel to Corollary 1, we can show immediately that

Corollary 2. *The map* $\mathfrak{C}[\Delta^n] \to c([n])$ *adjoint to the isomorphism* $\pi(\mathfrak{C}[\Delta^n]) \to [n]$ *given in Corollary 1 is a weak equivalence that is identity on objects.*

We also point out the following theorem:

Theorem 1. *Suppose* sSet *is endowed with the Joyal model structure, then the adjoint pair* $\mathfrak{C} \dashv \mathrm{N}$ *gives a Quillen equivalence between* sSet *and* Cat_Δ*.*

Remark 4. The theorem above indeed alludes to the fact that $\mathsf{Cat}_{\mathsf{sSet}}$ served as another model for ∞-categories other than sSet (or quasi-categories). As a reminder, there are indeed a number of classical ∞-categories constructed in this way: for example, the ∞-category of spaces S, the ∞-category of ∞-catgeories Cat_∞, derived ∞-category $\mathcal{D}_{\mathcal{A}}$ for abelian category $\mathcal{A}$, etc; of which we shall introduce the former two later.

However, one must have noted that this model is not that good, and people generally like to work with quasi-categories. For the reason, one could check this thread on overflow.

2.4 Explore $\mathrm{N}(\mathcal{C})$

In this subsection we briefly study the simplicial set $\mathrm{N}(\mathcal{C})$, where $\mathcal{C}$ is an arbitrary simplicially enriched category.

Note that $\mathfrak{C}[\Delta^0] = c([0])$, and that $\mathfrak{C}[\Delta^1] = c([1])$. We see that

$$\begin{aligned}\mathrm{N}(\mathcal{C})_0 &\coloneqq \mathrm{Hom}_{\mathsf{Cat}_\Delta}(\mathfrak{C}[\Delta^0], \mathcal{C}) \\ &= \mathrm{Hom}_{\mathsf{Cat}_\Delta}(c([0]), \mathcal{C}) \\ &\cong \mathrm{Hom}_{\mathsf{sSet}}([0], u\mathcal{C}),\end{aligned}$$

and similarly

$$\mathrm{N}(\mathcal{C})_1 \cong \mathrm{Hom}_{\mathsf{sSet}}([1], u\mathcal{C}).$$

Therefore, the objects of $\mathrm{N}(\mathcal{C})$ are given by objects of $\mathcal{C}$, and the morphisms of $\mathrm{N}(\mathcal{C})$ are given by the 1-morphisms of $\mathcal{C}$, that is, the objects of the hom-sets of $\mathcal{C}$.

We could also explore $\mathrm{N}(\mathcal{C})_2$. We omit the procedure (which is, however, worth of carrying out), and give the conclusion: the 2-simplices of $\mathrm{N}(\mathcal{C})$ are given by natural transformations $g \circ f \Rightarrow h$, where we have morphisms $x \xrightarrow{f} y$, $y \xrightarrow{g} z$, and $x \xrightarrow{h} z$, and its edges are g, f and $g \circ f$ respectively.

3 S and Cat_∞

As a corollary to Theorem 1, we see that for any Kan-enriched category $\mathcal{C}$, $\mathrm{N}(\mathcal{C})$ is an ∞-category. We also note that this is a proposition that can be proved directly. Besides, for any $\mathcal{C} \in \mathsf{Cat}_\Delta$, using Corollary 2, we can show that $\mathrm{N}(\mathcal{C})$ must be a composer. That is, any map $Sp^n \to \mathrm{N}(\mathcal{C})$ from the n-spine Sp^n to $\mathrm{N}(\mathcal{C})$ can be lifted to a map $\Delta^n \to \mathrm{N}(\mathcal{C})$.

Now, consider the category Kan of (small) Kan complexes. This category is Kan-enriched, whose Hom-set from $\mathcal{C}$ to $\mathcal{D}$ is given by the Kan complex $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, where $\mathcal{C}, \mathcal{D}$ are Kan complexes. We then define the ∞-catgeory of spaces $\mathsf{S} := \mathrm{N}(\mathsf{Kan})$.

Remark 5. The ∞-category S is really important: it serves as the role ”base space” in the setting of ∞-categories, as the category Set in ordinary category theory. For example, one can define the ∞-presheaf over an ∞-catgeory $\mathcal{C}$ as the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{S})$.

Remark 6. As is well-known, to each model category, one may, through localisation, associate a corresponding ∞-category. If one carries this procedure to sSet with the standard model structure, then one obtain the ∞-category S. Indeed, one has a more general theorem due to Dwyer–Kan:

Theorem 2 (Dwyer–Kan)**.** *Let $\mathcal{C}$ be a simplicial model category. Then the full subcategory of cofibrant-fibrant objects $\mathsf{Cat}_{\mathrm{cf}}$ is a Kan-enriched category. Furthermore, one has an equivalence between ∞-categories:*

$$L\mathcal{C} \simeq \mathrm{N}(\mathcal{C}_{\mathrm{cf}}),$$

where $L\mathcal{C}$ denotes the localisation of $\mathcal{C}$.

We then consider another example. Denote by QCat the Kan-enriched category of (small) ∞-categories. Its objects are (small) ∞-categories, and its Hom-set from ∞-category $\mathcal{C}$ to $\mathcal{D}$ is given by the maximal sub-∞-groupoid of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. We then define the ∞-category of ∞-categories $\mathsf{Cat}_\infty := \mathrm{N}(\mathsf{QCat})$.

Remark 7. **Warning!** One may think one can apply Theorem 2 to sSet with Joyal model struture. This is, however, not legistimate, as in this case sSet is not a simplical model category. However, if one consider the simplicial model category sSet^+ of marked simplicial sets, then Theorem 2 can apply and it turns out that the result is Cat_∞. Still, it is alluring to ask whether sSet with Joyal model struture localises to give Cat_∞.

Remark 8. One may ask why we take ”maximal sub-∞-groupoid” here. As far as the author knows, this is a matter of covenience. If we do not do so, we will then obtain the so-called $(\infty, 2)$-category of $(\infty, 1)$-categories. See the third section of [1].

References

[1] nLab authors. model structure for Cartesian fibrations. https://ncatlab.org/nlab/show/model+structure+for+Cartesian+fibrations, March 2024. Revision 28.